

# FRAC TAL WEYL LAWS FOR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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**ABSTRACT.** For asymptotically hyperbolic manifolds with hyperbolic trapped sets we prove a fractal upper bound on the number of resonances near the essential spectrum, with power determined by the dimension of the trapped set. This covers the case of general convex cocompact quotients (including the case of connected trapped sets) where our result implies a bound on the number of zeros of the Selberg zeta function in disks of arbitrary size along the imaginary axis. Although no sharp fractal lower bounds are known, the case of quasifuchsian groups, included here, is most likely to provide them.

Let  $M = \Gamma \backslash \mathbb{H}^n$ ,  $n \geq 2$ , be a convex cocompact quotient of hyperbolic space, i.e. a conformally compact manifold of constant negative curvature. Let  $\delta_\Gamma \in [0, n - 1)$  be the Hausdorff dimension of its limit set, which by Patterson–Sullivan theory equals the abscissa of convergence of its Poincaré series [Pa, Su79]. Let  $Z_\Gamma(s)$  be the Selberg zeta function:

$$Z_\Gamma(s) = \exp \left( - \sum_{\gamma \in \mathcal{P}} \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-sml(\gamma)}}{\det(\text{Id} - P_\gamma)} \right),$$

where  $\mathcal{P}$  is the set of primitive closed geodesics on  $M$ ,  $l(\gamma)$  is the length of  $\gamma$ , and  $P_\gamma$  is the Poincaré one-return map of  $\gamma$  in  $T^*M$ . The sums converge absolutely for  $\text{Re } s > \delta_\Gamma$  (so  $Z_\Gamma$  is nonvanishing there), and the function  $Z_\Gamma$  extends holomorphically to  $\mathbb{C} \setminus (-\mathbb{N}_0 \cup ((n - 1)/2 - \mathbb{N}))$  [Fr, BuOl, PaPe]. Let  $m_\Gamma(s)$  be the order of vanishing of  $Z_\Gamma$  at  $s$ .

**Theorem 1.** *For any  $R > 0$  there exists  $C > 0$  such that if  $t \in \mathbb{R}$ , then*

$$\sum_{|s-it| < R} m_\Gamma(s) \leq C(1 + |t|)^{\delta_\Gamma}. \quad (1.1)$$

This was proved by Guillopé–Lin–Zworski [GuLiZw] in the case when  $\Gamma$  is Schottky. In this paper we consider general convex cocompact quotients (see Figure 1 for examples), and, as we explain below, also give a generalization to the case of nonconstant curvature.

Theorem 1 follows from Theorem 2 below which holds in a general geometric setting. The novelty of our approach lies in combining recent results of Vasy [Va10] on effective meromorphic continuation with the technology of [SjZw] for resonance counting. (For a direct presentation of Vasy’s construction in the explicit setting of the hyperbolic cylinder, see Figure 2 and Appendix B.) A particular challenge comes from constructing Lyapunov/escape functions compatible with both approaches. We also simplify the counting argument on

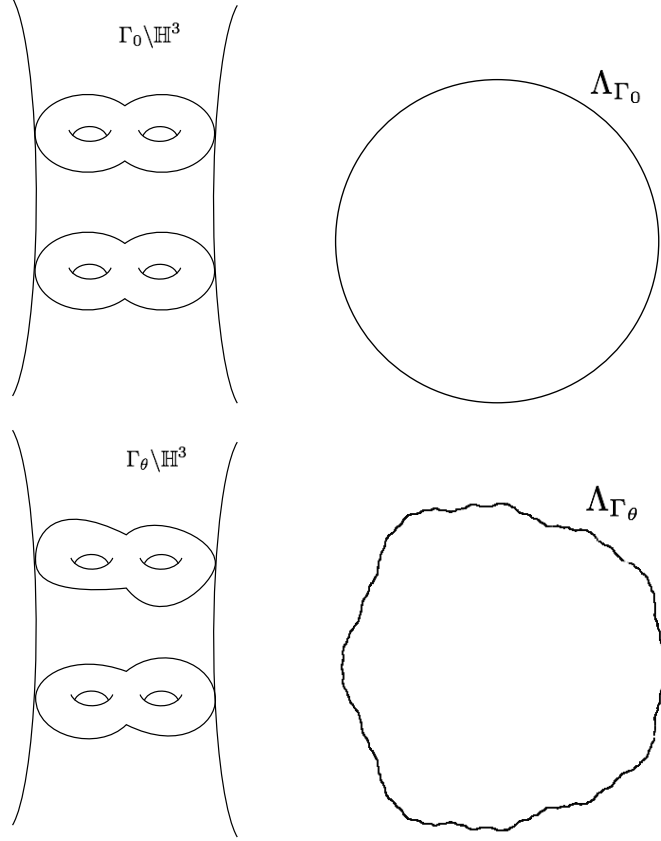


FIGURE 1. Let  $\Gamma_0$  be a cocompact Fuchsian group. Then  $\Gamma_0 \backslash \mathbb{H}^3$  is convex cocompact, its limit set  $\Lambda_{\Gamma_0}$  is a circle and  $\delta_{\Gamma_0} = 1$ . For  $\Gamma_\theta$  a *quasifuchsian bending* of  $\Gamma_0$ ,  $\Gamma_\theta \backslash \mathbb{H}^3$  is convex cocompact, but its limit set  $\Lambda_{\Gamma_\theta}$  is a *quasicircle* and  $\delta_{\Gamma_\theta} > 1$ . Theorems 1 and 2 apply, but  $\Gamma_\theta \backslash \mathbb{H}^3$  is not Schottky,  $\Lambda_{\Gamma_\theta}$  is connected, and the trapped set is of pure fractal dimension. See Appendix A.

$h$ -size scales by replacing the complicated second microlocalization of [SjZw] by suitably adapted functional calculus. The authors of [SjZw] also considered the use of functional calculus in their treatment of  $h$ -size neighborhoods of the energy surface, but chose a fully microlocal approach.

To state the main theorem, let  $(M, g)$  be an asymptotically hyperbolic manifold; i.e.  $M$  is the interior of a compact manifold with boundary  $\overline{M}$  and  $g$  is a metric on  $M$  such that

$$g = \frac{d\tilde{x}^2 + g_1}{\tilde{x}^2} \quad (1.2)$$

near  $\partial\overline{M}$ , where  $\tilde{x}$  is a defining function for  $\partial\overline{M}$  and  $g_1$  a 2-cotensor with  $g_1|_{\partial\overline{M}}$  a metric. Suppose  $g_1$  is even in the sense of being smooth in  $\tilde{x}^2$  (see also [Gu05a, Definition 1.2]).

Let  $\Delta_g$  be the positive Laplacian on  $(M, g)$ . Its resolvent,  $(\Delta_g - s(n-1-s))^{-1}$ , is meromorphic  $L^2(M) \rightarrow L^2(M)$  for  $\operatorname{Re} s > \frac{n-1}{2}$  and the essential spectrum is the line

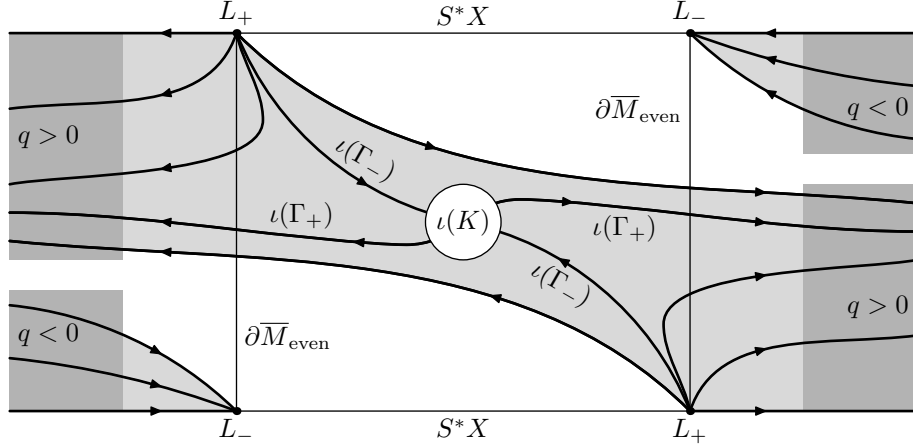


FIGURE 2. A schematic presentation of the dynamical complexity for the hyperbolic cylinder  $M = (-1, 1)_r \times \mathbb{S}_y^1$ . Here  $\iota(K)$  denotes the trapped set, which connects to infinity through the incoming and outgoing tails  $\iota(\Gamma_\pm)$ . The vertical lines  $\partial\overline{M}_{\text{even}}$  correspond to the two funnel ends  $\{r = \pm 1\}$ ; the horizontal variable is  $r$  and the vertical variable is the compactification  $\zeta/\langle\zeta\rangle$  of the momentum  $\zeta$  dual to  $r$ . The lighter shaded regions are the components  $\Sigma_\pm$  of the energy surface, with  $\Sigma_+$  the bigger region and  $\Sigma_-$  the union of two small ones. The darker shaded regions are the sets where the complex absorbing operator  $Q$  is elliptic. See Appendix B for more details.

$\operatorname{Re} s = \frac{n-1}{2}$ . As an operator from  $L_{\text{comp}}^2$  to  $L_{\text{loc}}^2$  it extends meromorphically to  $\mathbb{C}$ , with finite rank poles called *resonances*, as shown by Mazzeo–Melrose [MaMe] and Guillarmou [Gu05a] for asymptotically hyperbolic manifolds, Guillopé–Zworski [GuZw95a] when curvature near infinity is constant, and Vasy [Va10] for asymptotically hyperbolic manifolds with  $g_1$  even.

When  $M = \Gamma \backslash \mathbb{H}^n$ , Bunke–Olbrich [BuOl] and Patterson–Perry [PaPe, Theorems 1.5, 1.6] show that zeros of  $Z_\Gamma$  and *scattering poles* coincide on  $\mathbb{C} \setminus (-\mathbb{N}_0 \cup (\frac{n-1}{2} - \mathbb{N}))$ . Guillopé–Zworski [GuZw97] and Borthwick–Perry [BoPe, Theorem 1.1] show that scattering poles and resonances coincide off a discrete subset of  $\mathbb{R}$ . Below, we bound the density of resonances near the essential spectrum. Theorem 1 follows from the correspondence between resonances and zeros of  $Z_\Gamma$  (the discrete set where the correspondence fails, which has been further studied in e.g. [GrZw, Gu05b], is a subset of  $\mathbb{R}$  and hence irrelevant here).

We assume further that the geodesic flow on  $M$  is hyperbolic on its trapped set in the following sense of Anosov. (In fact, it is sufficient to assume hyperbolicity in the weaker sense of [Sj90, §5] and [SjZw, §7]. The latter includes the case of normally hyperbolic trapped sets, see for example [WuZw11].) Let  $p_0 \in C^\infty(T^*M)$  be the (shifted) geodesic Hamiltonian:

$$p_0(\rho) = g^{-1}(\rho, \rho) - 1,$$

where  $g^{-1}$  is the dual metric to  $g$ . Let  $H_{p_0}$  be the Hamilton vector field of  $p_0$  and  $\exp(tH_{p_0})$  its flow. Define the *trapped set* and its intersection with the energy surface  $p_0^{-1}(0)$  by

$$\tilde{K} = \{\rho \in T^*M \setminus 0 \mid \{\exp(tH_{p_0})\rho \mid t \in \mathbb{R}\} \text{ is bounded}\}, \quad K = \tilde{K} \cap p_0^{-1}(0).$$

Note that the homogeneity of  $p_0$  in the fibers implies that  $\tilde{K}$  is conic in the fibers, and that (1.2) implies that  $K$  is compact. Our assumption is that for any  $\rho \in K$ , the tangent space to  $p_0^{-1}(1)$  at  $\rho$  splits into flow, unstable, and stable subspaces [KaHa, Definition 17.4.1]:

- (1)  $T_\rho p_0^{-1}(1) = \mathbb{R}H_{p_0} \oplus E_\rho^+ \oplus E_\rho^-$ ,  $\dim E_\rho^\pm = n - 1$ ,
- (2)  $d\exp(tH_{p_0})E_\rho^\pm = E_{\exp(tH_{p_0})\rho}^\pm$  for all  $t \in \mathbb{R}$ ,
- (3) there exists  $\lambda > 0$  such that  $\|d\exp(tH_{p_0})v\| \leq Ce^{-\lambda|t|}\|v\|$  for all  $v \in E_\rho^\mp$ ,  $\pm t \geq 0$ .

Here we consider the differential  $d$  of  $\exp(tH_{p_0}): T^*M \rightarrow T^*M$  as a map  $T_\rho T^*M \rightarrow T_{\exp(tH_{p_0})\rho} T^*M$ .

Recall a bounded subset  $B$  of an  $N$ -dimensional manifold has upper Minkowski dimension

$$\inf\{d \mid \exists C > 0, \forall \varepsilon \in (0, 1], \text{Vol}(B_\varepsilon) \leq C\varepsilon^{N-d}\}, \quad (1.3)$$

where  $B_\varepsilon$  is the  $\varepsilon$ -neighborhood of  $B$ . The dimension is *pure* if the infimum is attained.

We state our main theorem for the semiclassical, nonnegative Laplacian with spectral parameter  $E = h^2 s(n - 1 - s) - 1$ , and define the multiplicity of a pole at  $E \in \mathbb{C}$  by

$$m_h(E) = \text{rank} \left[ \oint_E (h^2 \Delta_g - 1 - E')^{-1} dE': L_{\text{comp}}^2(M) \rightarrow L_{\text{loc}}^2(M) \right].$$

**Theorem 2** (Main theorem). *Let  $(M, g)$  be asymptotically hyperbolic in the sense of (1.2), and suppose  $g_1$  is even and the geodesic flow is hyperbolic on  $K$ . Let  $2\nu_0 + 1$  be the upper Minkowski dimension of  $K$ . For any  $\nu > \nu_0$ ,  $c_0 > 0$  there exist  $c_1, h_0 > 0$  such that*

$$\sum_{|E| < c_0 h} m_h(E) \leq c_1 h^{-\nu}, \quad (1.4)$$

for  $h \in (0, h_0]$ . If  $K$  is of pure dimension, we may take  $\nu = \nu_0$ .

Our assumptions hold for  $M$  convex cocompact, but they only concern the asymptotic structure of  $g$  at infinity, and the dynamics of the geodesic flow at  $K$ . The latter assumption holds when  $M$  has (possibly variable) negative curvature [KL, Theorem 3.9.1].

When  $M$  is convex cocompact, we have  $\dim K = 2\delta_\Gamma + 1$ , and the Minkowski dimension is pure and equals the Hausdorff dimension ([Su79, §3], see also [Ni, §8.1], [BiJo, Corollary 1.5]). Theorem 1 follows from Theorem 2 and the fact that resonances of the Laplacian and zeros of the Selberg zeta function agree with multiplicities in the domains we study.

Theorem 2 is to be compared with the Weyl law for eigenvalues of a compact manifold:

$$\sum_{|E| < a} m_h(E) = (2\pi h)^{-n} \text{Vol}(p_0^{-1}[-a, a]) + \mathcal{O}(h^{-n+1}), \quad a \in (0, 1), \quad (1.5)$$

where  $n = \dim M$ , and with the corresponding bound when one considers smaller domains

$$\sum_{|E| < c_0 h} m_h(E) \leq c_1 h^{-n+1}, \quad (1.6)$$

see for example [SjZw, (1.1)]. The notation in (1.5) and (1.6) is as in (1.4) but the manifold is compact, so  $m_h(E) \neq 0$  only for real  $E$  and gives the multiplicity of the eigenvalue at  $E$ .

If  $\Gamma$  is cocompact Fuchsian, as in Figure 1, the Laplacian on  $\Gamma \backslash \mathbb{H}^3$  is unitarily equivalent to  $\bigoplus_{j \in \mathbb{N}_0} D_r^2 + \lambda_j \operatorname{sech}^2 r + 1$ , where  $\lambda_j \geq 0$  are the eigenvalues of  $\Gamma \backslash \mathbb{H}^2$ . By [GuZw95b, Appendix], the scattering poles of  $\Gamma \backslash \mathbb{H}^3$  are  $s_{j,k} = \sqrt{1/4 - \lambda_j} + 1/2 - k$  where  $j \in \mathbb{N}$ ,  $k \in \mathbb{N}_0$ , and the square root has values in  $(-1/2, 0] \cup i\mathbb{R}$ . The Weyl law for  $\Gamma \backslash \mathbb{H}^2$  implies that in this case (1.1) and (1.4) can be improved to asymptotics:

$$\sum_{|s-it| < R} m_\Gamma(s) = C|t|^{\delta_\Gamma}(1 + o(1)), \quad \sum_{|E| < c_0 h} m_h(E) = c_1 h^{-\nu}(1 + o(1)), \quad \delta_\Gamma = \nu = 1.$$

The first polynomial upper bounds on the resonance counting function are due to Melrose [Me83], and the first bounds involving geometric data (the dimension of the trapped set) are due to Sjöstrand [Sj90]. Sjöstrand's result was proved for convex cocompact surfaces by Zworski [Zw99] (with an improvement by Naud [Na]) and for analytic scattering manifolds by Wunsch–Zworski [WuZw00]. Guillopé–Lin–Zworski [GuLiZw] proved Theorem 2 for convex cocompact Schottky groups, including general convex cocompact surfaces. Sjöstrand–Zworski [SjZw] obtained the first result allowing  $C^\infty$  perturbations, and our analysis of the trapped set partially follows theirs, with modifications needed due to the asymptotically hyperbolic infinity. As in [GuLiZw] and [SjZw], we count resonances in  $\mathcal{O}(h)$  size regions, rather than in the larger regions of [Sj90]. Most recently, Nonnenmacher–Sjöstrand–Zworski [NoSjZw1, NoSjZw2] studied general topologically one-dimensional hyperbolic flows and proved the analog of Theorem 2 for scattering by several convex obstacles.

Much less is known about lower bounds. In [GéSj], Gérard–Sjöstrand studied semiclassical problems with  $K$  a single periodic orbit and proved that resonances lie asymptotically on a lattice, implying in particular sharp upper and lower bounds with  $\nu = 0$ . Similar results hold for spherically symmetric problems, as studied by Sá Barreto–Zworski [SáZw], and (under a stronger hyperbolicity condition) for perturbations of them, as studied by the second author [Dy12a]. Most recently, Weyl asymptotics for resonances in strips have been obtained under more restrictive dynamical assumptions in work in preparation by Faure–Tsuji [FaTs] and the second author [Dy12b]. Nonnenmacher–Zworski [NoZw] proved asymptotics with fractal  $\nu$  for toy models of open quantum systems, and Lu–Sridhar–Zworski [LuSrZw] give numerical evidence supporting an asymptotic in the case of obstacle scattering. Jakobson–Naud [JaNa] proved logarithmic lower bounds with exponent related to  $\delta_\Gamma$  for convex cocompact surfaces, and in the arithmetic case their bounds are fractal, but the power is different from the one for the upper bound.

For a broader introduction to the subject of the distribution of resonances for systems with hyperbolic classical dynamics, we refer the reader to the recent review paper of Nonnenmacher [No], which describes many of these and other results. It also includes a discussion of the theoretical and experimental physics literature on the subject, which supports the optimality of our upper bound (although without rigorous proofs).

An interesting open problem is to prove the analog of Theorem 1 or 2 for manifolds with cusps; when  $M = \Gamma \backslash \mathbb{H}^n$  this means  $\Gamma$  has parabolic elements. If cusps have mixed rank the problem is harder; Guillarmou–Mazzeo [GuMa] show that the resolvent is meromorphic in  $\mathbb{C}$  but continuation of the zeta function is not known. If  $n = 2$  all cusps have full rank, and the main difficulty comes from the fact that  $K$  extends into the cusp and is not compact.

**Outline of the proof of Theorem 2.** (See §3 for a review of semiclassical notation and terminology.) We begin with Vasy’s construction [Va10, Va11] of a Fredholm semiclassical pseudodifferential operator  $P(z) - iQ$ , on a compact manifold  $X$ , with  $M$  diffeomorphic to an open subset of  $X$ , and such that the resonances of the semiclassical Laplacian on  $M$  are a subset of the poles of  $(P(z) - iQ)^{-1}$ ; here  $1 + E = h^2(n - 1)^2/4 + (1 + z)^2$ . The operator  $Q$  is supported away from  $\overline{M} \subset X$ , and  $P(z)$  is a differential operator such that  $P(z)|_M$  is  $\Delta_g$  conjugated and weighted (see (4.1)). We review this construction in §4.1 and summarize the main results in Lemma 2.1; see Figure 2 for the global dynamics of the corresponding Hamiltonian system. We will show that for any  $C_0 > 0$  there is  $h_0 > 0$  such that  $(P(z) - iQ)^{-1}$  has  $\mathcal{O}(h^{-\nu})$  many poles in  $\{|\operatorname{Re} z| \leq C_0 h, |\operatorname{Im} z| \leq C_0 h\}$  for  $h \in (0, h_0]$ .

To do this we introduce the conjugated operator

$$P_t(z) = e^{-tF} T_s (P(z) - iQ) T_s^{-1} e^{tF},$$

where  $t > 0$  is a large parameter (independent of  $h$ ) and  $T_s, F$  are pseudodifferential operators on  $X$ , with  $F$  in an exotic calculus. We will show that this operator is invertible up to a remainder of finite rank  $\mathcal{O}(h^{-\nu})$ , and then conclude using Jensen’s formula (§2). This follows from the estimate

$$\|u\|_{H_h^{1/2}(X)} \leq Ch^{-1} \|(P_t(z) - ithA)u\|_{H_h^{-1/2}(X)}, \quad (1.7)$$

where  $A = A_R + A_E$ , with  $A_R$  of rank  $\mathcal{O}(h^{-\nu})$  and  $\|A_E\|_{H_h^{1/2} \rightarrow H_h^{-1/2}} = \mathcal{O}(\tilde{h})$ , where  $0 < \tilde{h} \ll 1$  depends on  $t$  but not  $h$ . This, (1.7), is the main estimate of the paper (see (2.2)).

To prove (1.7), we begin with the Taylor expansion

$$P_t(z) = T_s(P(z) - iQ)T_s^{-1} + t[T_s(P(z) - iQ)T_s^{-1}, F] + \mathcal{O}_t(h\tilde{h}), \quad (1.8)$$

(this is (7.5)). We then use a microlocal partition of unity (Lemma 7.3) to divide the phase space  $\overline{T}^*X$  into regions where the principal symbols of the various terms are elliptic. We construct  $T_s, F$ , and  $A$  such that these regions cover  $\overline{T}^*X$ , after which we prove (1.7) using a positive commutator argument. In fact, because  $P_t(z) - ithA = P(z) - iQ + \mathcal{O}(h \log(1/h))$ ,

it suffices to check that the intersection  $\{\langle \xi \rangle^{-2}p = 0\} \cap \{\langle \xi \rangle^{-2}q = 0\}$  of the characteristic sets of  $P(0)$  and  $Q$  is covered.

More specifically,  $T_s$  is an elliptic pseudodifferential operator whose order  $s$  is large enough that the principal symbol of  $h^{-1} \operatorname{Im} T_s(P(z) - iQ)T_s^{-1}$  is elliptic near the intersection of  $\{\langle \xi \rangle^{-2}p = 0\}$  with the fiber infinity of  $\bar{T}^*X$ . We include a neighborhood of the *radial points* of  $P(z)$  (i.e. the fixed points of its symbol's Hamiltonian vector field) and this (§4.2) is where our analysis near the spatial infinity of  $M$  is different from that of [Va10, Va11]. The ellipticity of the principal symbol of  $h^{-1} \operatorname{Im} T_s(P(z) - iQ)T_s^{-1}$  (with a favorable sign) is proved in Lemma 4.6, and it is used in the positive commutator argument in Lemma 7.5.

The operator  $F$  has the form

$$F = \widehat{F} + M \log(1/h)F_0,$$

where  $M > 0$  and  $F_0, \widehat{F}$  are quantizations of *escape functions*, that is functions monotonic along bicharacteristic flowlines of  $P(0)$  in certain regions of  $\{\langle \xi \rangle^{-2}p = 0\}$ . For  $F_0$  this monotonicity holds outside of a fixed neighborhood of the radial points and of the trapped set  $\iota(K)$ , (here  $\iota: T^*M \rightarrow \bar{T}^*X$  is the slightly modified inclusion map defined in (4.6); note that no function can be monotonic along bicharacteristic flowlines at the radial points or at  $\iota(K)$ ) giving ellipticity of the principal symbol of  $h^{-1} \operatorname{Im}[T_s(P(z) - iQ)T_s^{-1}, F_0]$  in that region. For  $\widehat{F}$  this monotonicity holds at points in a fixed neighborhood of  $\iota(K)$  which lie at least  $\sim (h/\tilde{h})^{1/2}$  away from  $\iota(K)$  (recall  $\tilde{h}$  is small but independent of  $h$ )<sup>1</sup>. We need the monotonicity up to a neighborhood of  $\iota(K)$  which is as small as possible so that the correction term  $A$ , needed for the global estimate (1.7), can be microsupported in a small enough set that  $A$  is of rank  $\mathcal{O}(h^{-\nu})$  plus a small remainder. The escape function for  $\widehat{F}$  is taken directly from [SjZw], and it is here that we use the assumption that the geodesic flow is hyperbolic on  $K$ . The escape functions are constructed in Lemmas 4.7 and 7.1.

The operator  $A$  has the form

$$A = \chi((h/\tilde{h})\widehat{P})\widetilde{A}, \tag{1.9}$$

where  $\widehat{P}$  is an elliptic self-adjoint operator whose principal symbol agrees with that of  $P(0)$  near  $\iota(K)$ ,  $\chi \in C_0^\infty(\mathbb{R})$  is 1 near 0, and  $\widetilde{A}$  is microsupported in a neighborhood of size  $\mathcal{O}(h/\tilde{h})^{1/2}$  of  $\iota(K)$ . The operator  $\widetilde{A}$  is pseudodifferential in the exotic class of §5.1, but  $\chi((h/\tilde{h})\widehat{P})$  is not even pseudolocal: it propagates semiclassical singularities along bicharacteristics of  $\widehat{P}$  (which near the microsupport of  $\widetilde{A}$  are the same as bicharacteristics of  $P(0)$ ). This type of operator is treated in [SjZw] using a *second microlocal* pseudodifferential calculus. In §5.2 we use a more basic approach based on the Fourier inversion formula,

$$\chi((\tilde{h}/h)\widehat{P}) = \frac{1}{2\pi} \int \hat{\chi}(t) e^{it(h/\tilde{h})\widehat{P}} dt,$$

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<sup>1</sup>We need  $\tilde{h}$  because if a symbol's derivatives grow like  $h^{-1/2}$ , then its quantization does not have an asymptotic expansion in powers of  $h$ . As we show in §5.1, including  $\tilde{h}$  gives an expansion in powers of  $\tilde{h}$ .



and functional calculus to prove only the few microlocal properties of  $\chi((\tilde{h}/h)\hat{P})$  we need. We use them in the positive commutator argument in Lemma 7.8, and we show that  $A = A_R + A_E$ , with  $\text{rank } A_R = \mathcal{O}(h^{-\nu})$  and  $\|A_E\| = \mathcal{O}(\tilde{h})$ , in Lemma 6.1 and §7.4.

### Structure of the paper.

- In §2 we state the main properties of the extended manifold  $X$  and the modified Laplacian  $P(z) - iQ$  from [Va10, Va11] in Lemma 2.1 and then use Jensen's formula to reduce the proof of Theorem 2 to the main lemma, Lemma 2.2.
- In §3 we review the notation used in the paper and properties of semiclassical pseudodifferential and Fourier integral operators.
- In §4.1 we review the construction of  $X$  and  $P(z) - iQ$  and the proof of Lemma 2.1. In §4.2 we introduce the conjugation by  $T_s$  and prove estimates near the radial points, and then define the escape function  $f_0$  used to separate the analysis near infinity from the analysis near the trapped set.
- In §5.1 we review the part of the exotic  $\Psi_{1/2}$  calculus of [SjZw] needed here. In §5.2 we study microlocal properties of operators of the form  $\chi((h/\tilde{h})\hat{P})$  as in (1.9).
- In §6 we prove that operators of the form of  $A$  in (1.9) can be written  $A = A_R + A_E$ , with  $\|A_E\|_{L^2 \rightarrow L^2} = \mathcal{O}(\tilde{h})$  and  $\text{rank } A_R = \mathcal{O}_{\tilde{h}}(h^{-\nu})$ .
- In §7 we prove Lemma 2.2. In §7.1 we use the results of §§3–5 to prove positive commutator estimates for the modified conjugated operator  $P_t(z) - ithA$ . In §7.2 and §7.3 we use these to prove semiclassical resolvent estimates, and in §7.4 we apply the results of §6 to the operator  $A$  from (1.9).
- In Appendix A we construct the quasifuchsian group in Figure 1.
- In Appendix B we give a direct presentation of Vasy's construction from §4.1, and of the escape functions from Lemmas 4.7 and 7.1, when  $M$  is the hyperbolic cylinder.

For the convenience of the reader interested in our approach to second microlocalization but not in the analysis near infinity, §5.2 is independent of §4 (as are §5.1 and §6).

## 2. PROOF OF THEOREM 2

We start by reviewing in Lemma 2.1 Vasy's recent description [Va11] of the scattering resolvent and resonances on an asymptotically hyperbolic even space  $(M, g)$ ; this allows us to replace the Laplacian on  $M$  by a Fredholm pseudodifferential operator  $P(z) - iQ$  on a compact manifold  $X$ , adapted to proving semiclassical estimates. See §4.1 for details.

**Lemma 2.1.** *Assume that  $(M, g)$  is asymptotically hyperbolic and even. Then there exist a compact manifold  $X$  without boundary, an order 2 semiclassical differential operator  $P(z)$  depending holomorphically on  $z$ , and an order 2 semiclassical pseudodifferential operator  $Q$  depending holomorphically on  $z$  such that for  $h$  small enough,*



- (1) for  $\text{Im } z > -C_0 h$  and  $s > C_0$ , the family of operators

$$P(z) - iQ : \{u \in H_h^{s+1/2}(X) \mid P(0)u \in H_h^{s-1/2}(X)\} \rightarrow H_h^{s-1/2}(X)$$

is Fredholm of index zero, and a preimage of a smooth function under this operator is again smooth. Here the domain is equipped with the norm  $\|u\| = \|u\|_{H_h^{s+1/2}} + \|P(0)u\|_{H_h^{s-1/2}}$ . The inverse

$$(P(z) - iQ)^{-1} : H_h^{s-1/2}(X) \rightarrow H_h^{s+1/2}(X)$$

is meromorphic in  $z \in \{\text{Im } z > -C_0 h\}$  with poles of finite rank;

- (2) the set of poles of  $(P(z) - iQ)^{-1}$  in  $\{\text{Im } z > -C_0 h\}$  contains (including multiplicities) the set of poles of the continuation of

$$(h^2(\Delta_g - (n-1)^2/4) - (z+1)^2)^{-1} : L_{\text{comp}}^2(M) \rightarrow L_{\text{loc}}^2(M)$$

from  $\{\text{Im } z > 0\}$  to  $\{\text{Im } z > -C_0 h\}$ .

To prove Theorem 2 we will apply Lemma 2.1 (and Lemma 2.2 below) with  $C_0$  a large constant multiple of  $c_0$ . Throughout the paper we will work in the domain  $\{|\text{Re } z| \leq C_0 h, \text{Im } z \geq -C_0 h\}$ . In the main lemma, we define a modified, conjugated operator  $\tilde{P}_t(z)$ , and prove semiclassical estimates for it. See Section 3.1 for the semiclassical notation.

**Lemma 2.2.** (Main lemma) Assume that  $(M, g)$  and  $\nu$  satisfy the assumptions of Theorem 2. Let  $X, P(z), Q, C_0$  be as in Lemma 2.1, and let  $T_s \in \Psi^s(X)$  be any (semiclassically) elliptic operator. We introduce a parameter  $\tilde{h} > 0$ ; the estimates below hold for  $\tilde{h}$  small enough and  $h$  small enough depending on  $\tilde{h}$ , and the constants in these estimates are independent of  $\tilde{h}, h$ , and  $z$  in the specified range, except for  $C(\tilde{h})$  below.

Then there exist  $t > 0$  and compactly microlocalized polynomially bounded operators  $A, F$  depending on  $\tilde{h}$ , with  $e^{\pm tF} - 1$  polynomially bounded and compactly microlocalized and

- (1) the modified conjugated operator

$$\tilde{P}_t(z) = e^{-tF} T_s (P(z) - iQ) T_s^{-1} e^{tF} - i t h A \quad (2.1)$$

satisfies the estimate

$$|\text{Re } z| \leq C_0 h, \quad |\text{Im } z| \leq C_0 h, \quad u \in C^\infty(X) \implies \|u\|_{H_h^{1/2}} \leq C h^{-1} \|\tilde{P}_t(z)u\|_{H_h^{-1/2}}; \quad (2.2)$$

- (2) if  $\tilde{h}, \varepsilon$  are small enough and  $h$  is small enough depending on  $\tilde{h}, \varepsilon$ , then we have the improved estimate in the upper half-plane:

$$|\text{Re } z| \leq C_0 h, \quad C_0 h \leq \text{Im } z \leq \varepsilon, \quad u \in C^\infty(X) \implies \|u\|_{H_h^{1/2}} \leq \frac{C}{\text{Im } z} \|\tilde{P}_t(z)u\|_{H_h^{-1/2}}; \quad (2.3)$$

- (3) we can write  $A = A_R + A_E$ , where  $A_R, A_E$  are compactly microlocalized and for some constant  $C(\tilde{h})$  independent of  $h$ ,

$$\|A_R\|_{H_h^{1/2} \rightarrow H_h^{-1/2}} = \mathcal{O}(1), \quad \|A_E\|_{H_h^{1/2} \rightarrow H_h^{-1/2}} = \mathcal{O}(\tilde{h}), \quad \text{rank } A_R \leq C(\tilde{h}) h^{-\nu}.$$

The conjugations by  $T_s$  and  $F$  modify  $P(z) - iQ$  to make it semiclassically elliptic away from the trapped set, without disturbing the poles of the resolvent. The parameter  $t$  is a coupling constant for the positive commutator term  $t[T_s(P(z) - iQ)T_s^{-1}, F]$  (see (1.8) above and see §7.1 below for details), taken to be large enough that it overcomes the error term arising from the unfavorable sign of  $\operatorname{Im} z$ . The correction term  $-ithA$  (which makes  $\tilde{P}_t(z)$  invertible) is compactly microlocalized in a small enough neighborhood of the trapped set in the energy surface that it can be approximated by an operator of rank  $\mathcal{O}(h^{-\nu})$ ; it does affect the poles of the resolvent, but as we will see below it can remove no more than  $\mathcal{O}(h^{-\nu})$  of them in the set  $\{|\operatorname{Re} z| < C_0 h/2, |\operatorname{Im} z| < C_0 h/2\}$ .

*Proof of Theorem 2 assuming Lemmas 2.1 and 2.2.* We follow [SjZw, §6.1]. Fix  $s > C_0$ . For  $\operatorname{Im} z \geq -C_0 h$ , the operators

$$P_t(z) = e^{-tF} T_s(P(z) - iQ) T_s^{-1} e^{tF}$$

and  $\tilde{P}_t(z)$  are Fredholm of index zero

$$\{u \in H_h^{1/2} \mid P(0)u \in H_h^{-1/2}\} \rightarrow H_h^{-1/2}.$$

For  $P_t(z)$ , this follows immediately from Lemma 2.1 as  $e^{tF} - 1$  is compactly microlocalized and thus  $e^{tF}$  preserves Sobolev spaces, while  $T_s^{-1}$  maps  $H_h^{-1/2} \rightarrow H_h^{s-1/2}$  and  $H_h^{s+1/2} \rightarrow H_h^{1/2}$ . (Note however that the norm of  $e^{tF}$  grows as  $h \rightarrow 0$ .) For  $\tilde{P}_t(z)$ , we additionally use that  $A$  is compactly microlocalized and thus a compact perturbation of  $P_t(z)$ .

Let  $\varepsilon > 0$  be a small constant and consider the rectangle

$$R_0 = \{\operatorname{Re} z \in [-C_0 h, C_0 h], \operatorname{Im} z \in [-C_0 h, \varepsilon]\}.$$

It follows from parts 1 and 2 of Lemma 2.2 that  $\tilde{P}_t(z)^{-1}$  has no poles in  $R_0$  and satisfies

$$z \in R_0 \implies \|\tilde{P}_t(z)^{-1}\|_{H_h^{-1/2} \rightarrow H_h^{1/2}} \leq \frac{C}{\max(h, \operatorname{Im} z)}.$$

We now use the decomposition  $A = A_R + A_E$  from Lemma 2.2(3). Since  $\|A_E\| = \mathcal{O}(\tilde{h})$ , for  $\tilde{h}$  small enough  $\|(-ithA_E)\tilde{P}_t(z)^{-1}\|_{H_h^{-1/2} \rightarrow H_h^{1/2}} \leq 1/2$  and we get

$$z \in R_0 \implies \|(P_t(z) - ithA_R)^{-1}\|_{H_h^{-1/2} \rightarrow H_h^{1/2}} \leq \frac{C}{\max(h, \operatorname{Im} z)}. \quad (2.4)$$

Now,

$$(P_t(z) - ithA_R)^{-1} P_t(z) = 1 + ith(P_t(z) - ithA_R)^{-1} A_R =: 1 + K(z), \quad z \in R_0.$$

Therefore, the poles of  $P_t(z)^{-1}$  (and hence of  $(P(z) - iQ)^{-1}$ ) are contained, including multiplicities, in the zeros of

$$k(z) = \det(1 + K(z));$$

indeed, see [Sj02, Proposition 5.16] for a general statement and see [Zw, §D.1] for a discussion of the theory of Grushin problems which is used there.

By Lemma 2.2(3),  $K(z)$  has norm  $\mathcal{O}(1)$  and rank  $\mathcal{O}(h^{-\nu})$ . Therefore,

$$|k(z)| \leq e^{Ch^{-\nu}}, \quad z \in R_0. \quad (2.5)$$

Since  $\|ithA_R\| = \mathcal{O}(h)$ , we have  $\|K(z_0)\| \leq 1/2$  for  $z_0 = iC_1h$  with  $C_1 > 0$  large enough, so

$$|k(z_0)| \geq e^{-Ch^{-\nu}}. \quad (2.6)$$

Define rectangles  $R_2 \subset R_1 \subset R_0$  by

$$\begin{aligned} R_1 &= \{\operatorname{Re} z \in (-C_0h, C_0h), \operatorname{Im} z \in (-C_0h, 4C_1h)\}, \\ R_2 &= \{\operatorname{Re} z \in (-C_0h/2, C_0h/2), \operatorname{Im} z \in (-C_0h/2, 2C_1h)\}. \end{aligned}$$

Then the estimate (2.5) holds in  $R_1$ , while  $z_0 \in R_2$ . By the Riemann mapping theorem, there exists a unique conformal map  $w(z)$  from  $R_1$  onto the ball  $B_1 = \{|w| < 1\}$  with  $w(z_0) = 0$  and  $w'(z_0) \in \mathbb{R}^+$ . We then see that  $w(R_2) \subset B_2 = \{|w| < 1 - \delta\}$  for some  $\delta > 0$  independent of  $h$  (as  $h^{-1}R_1$  and  $h^{-1}R_2$  do not depend on  $h$ ). One can now apply Jensen's formula (see for example [Ti, §3.61, equation (2)]) to the function  $k_1(w) = k(z(w))$  on  $B_1$ : if  $n(r)$  is the number of zeros of  $k_1$  in  $\{|w| < r\}$ , then

$$\int_0^{1-\delta/2} \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |k_1((1-\delta/2)e^{i\theta})| d\theta - \log |k_1(0)| \leq 2Ch^{-\nu}.$$

Therefore,

$$n(1-\delta) \leq \frac{2}{\delta} \int_{1-\delta}^{1-\delta/2} \frac{n(r)}{r} dr = \mathcal{O}(h^{-\nu}).$$

This estimates the number of zeroes of  $k_1(w)$  in  $B_2$ , thus the number of zeroes of  $k(z)$  in  $R_2$ , and thus the number of resonances in  $R_2$ , as needed.  $\square$

### 3. SEMICLASSICAL PRELIMINARIES

**3.1. Notation and pseudodifferential operators.** In this section, we review certain notions of semiclassical analysis; for a comprehensive introduction to this area, the reader is referred to [Zw] or [DiSj]. We will use the notation of [Va11, §2], with some minor changes. Consider a (possibly noncompact) manifold  $X$  without boundary. Following [Me94], the symbols we consider will be defined on the *fiber-radial compactification*  $\overline{T}^*X$  of the cotangent bundle; its boundary, called the fiber infinity, is associated with the spherical bundle  $S^*X$  and its interior is associated with  $T^*X$ . Denote by  $(x, \xi)$  a typical element of  $\overline{T}^*X$ ; here  $x \in X$  and  $\xi$  is in the radial compactification of  $T_x^*X$ . We fix a smooth inner product on the fibers of  $T^*X$ ; if  $|\cdot|$  is the norm on the fibers generated by this inner product, let

$$\langle \xi \rangle = (1 + |\xi|^2)^{1/2}.$$

Then  $\langle \xi \rangle^{-1}$  is a boundary defining function on  $\overline{T}^*X$ . (The smooth structure of  $\overline{T}^*X$  is independent of the choice of the inner product.)

Let  $k \in \mathbb{R}$ . A smooth function  $a(x, \xi)$  on  $T^*X$  is called a classical symbol of order  $k$ , if  $\langle \xi \rangle^{-k}a$  extends to a smooth function on  $\overline{T}^*X$ . We denote by  $S_{\text{cl}}^k(X)$  the algebra

of all classical symbols. If  $a$  also depends on the semiclassical parameter  $h > 0$ , it is called a classical semiclassical symbol of order  $k$ , if there exists a sequence of functions  $a_j(x, \xi) \in S_{\text{cl}}^{k-j}(X)$ ,  $j = 0, 1, \dots$ , such that  $a \sim \sum_j h^j a_j$  in the following sense: for each  $J$ , the function  $h^{-J} \langle \xi \rangle^{J-k} (a - \sum_{j < J} h^j a_j)$  extends to a smooth function on  $\overline{T^*X} \times [0, h_0)$ , for  $h_0 > 0$  small enough. In this case,  $a_0$  is called the (semiclassical) *principal part* of  $a$ : it captures  $a$  to leading order in both  $h$  and  $\langle \xi \rangle^{-1}$ . The semiclassical symbol  $a$  is classical if and only if  $\tilde{a} = \langle \xi \rangle^{-k} a$  extends to a smooth function on  $\overline{T^*X} \times [0, h_0)$ , and for each differential operator  $\partial^j$  of order  $j$  on  $\overline{T^*X}$ , the restriction of  $\partial^j \tilde{a}$  to  $S^*X$  is a polynomial of degree no more than  $j$  in  $h$ .

For real-valued  $a \in S_{\text{cl}}^k(X)$ , we denote by  $H_a$  the Hamiltonian vector field generated by  $a$  with respect to the standard symplectic form on  $T^*X$ . Then  $\langle \xi \rangle^{1-k} H_a$  can be extended to a smooth vector field on  $\overline{T^*X}$  and this extension preserves the fiber infinity  $S^*X$ .

The class  $S_{h,\text{cl}}^k(X)$  of classical semiclassical symbols is closed under the standard operations of semiclassical symbol calculus (multiplication, adjoint, change of coordinates); therefore, one can consider the algebra  $\Psi^k(X)$  of semiclassical pseudodifferential operators with symbols in  $S_{h,\text{cl}}^k(X)$ . (See [Zw, Chapters 4, 9, and 14] for more information.) We do not use the  $\Psi_{\text{cl}}^k(X)$  notation, as we will only operate with classical operators until §5.1, where a different class will be introduced. We require that all elements of  $\Psi^k$  be properly supported operators, so that they act  $C^\infty(X) \rightarrow C^\infty(X)$  and  $C_0^\infty(X) \rightarrow C_0^\infty(X)$ ; in particular, we can multiply two such operators. If  $H_h^s(X)$  denotes the semiclassical Sobolev space of order  $s$  (recall that when  $X$  is compact, one choice of norm is  $\|u\|_{H_h^s(X)} = \|(1 + h^2 \Delta_X)^{s/2} u\|_{L^2(X)}$ , where  $\Delta_X$  is the Laplacian on  $X$ ), then each  $A \in \Psi^k$  is bounded  $H_{h,\text{comp}}^s(X) \rightarrow H_{h,\text{comp}}^{s-k}(X)$  uniformly in  $h$ . Here  $H_{h,\text{comp}}^s$  consists of compactly supported distributions lying in  $H_h^s$ ; in this article, we will mostly work on compact manifolds, where this space is the same as  $H_h^s$ .

For  $A \in \Psi^k$ , the principal part  $\sigma(A)$  of the symbol of  $A$  is independent of the quantization procedure. We call  $\sigma(A)$  the *principal symbol* of  $A$ , and  $\sigma(A) = 0$  if and only if  $A \in h\Psi^{k-1}$ . We remark that  $\sigma(A)$  is the restriction of the standard semiclassical principal symbol on the larger standard semiclassical algebra. The principal symbol enjoys the multiplicativity property  $\sigma(AB) = \sigma(A)\sigma(B)$  and the commutator identity  $\sigma(h^{-1}[A, B]) = -i\{\sigma(A), \sigma(B)\}$ , where  $\{\cdot, \cdot\}$  is the Poisson bracket.

We define the (closed) semiclassical *wavefront set*  $\text{WF}_h(A) \subset \overline{T^*X}$  as follows: if  $a$  is the full symbol of  $A$  in some quantization, then  $(x_0, \xi_0) \notin \text{WF}_h(A)$  if and only if there exists a neighborhood  $U$  of  $(x_0, \xi_0, h = 0)$  in  $\overline{T^*X} \times [0, h_0)$  such that for each  $N$ ,  $h^{-N} \langle \xi \rangle^N a$  is smooth in  $U$ . Since the operations of semiclassical symbol calculus are defined locally modulo  $\mathcal{O}(h^\infty \langle \xi \rangle^{-\infty})$ , this definition does not depend on the choice of quantization, and we also have  $\text{WF}_h(AB) \subset \text{WF}_h(A) \cup \text{WF}_h(B)$  and  $\text{WF}_h(A^*) = \text{WF}_h(A)$ . We say that  $A = B$  *microlocally* in some set  $V \subset \overline{T^*X}$  if  $\text{WF}_h(A - B) \cap V = \emptyset$ . If  $\text{WF}_h(A)$  is a compact subset of  $T^*X$ , and in particular does not intersect the fiber infinity  $S^*X$ , then we call  $A$  *compactly microlocalized*. Denote by  $\Psi^{\text{comp}}(X)$  the class of all compactly

microlocalized pseudodifferential operators; these operators lie in  $\Psi^k(X)$  for all  $k$ . Note that for a noncompact  $X$ , compactly microlocalized operators need not have compactly supported Schwartz kernels.

The wavefront set of  $A \in \Psi^k$  is empty if and only if  $A \in h^N \Psi^{-N}$  for all  $N$ . In this case, we write  $A = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ . This property can also be stated as follows:  $A$  is properly supported, smoothing, and each  $C^\infty$  seminorm of its Schwartz kernel is  $\mathcal{O}(h^\infty)$ . With this restatement,  $B = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$  makes sense even for  $B : \mathcal{E}'(X_2) \rightarrow C^\infty(X_1)$  with  $X_1 \neq X_2$ .

We now explain our use of the  $\mathcal{O}(\cdot)$  notation. For an operator  $A$ , we write

$$A = \mathcal{O}_Z(f)_{\mathcal{X} \rightarrow \mathcal{Y}}, \quad (3.1)$$

where  $\mathcal{X}, \mathcal{Y}$  are Hilbert spaces,  $Z$  is some list of parameters, and  $f$  is an expression depending on  $h$  and perhaps some other parameters, if the  $\mathcal{X} \rightarrow \mathcal{Y}$  operator norm of  $A$  is bounded by  $Cf$ , where  $C$  depends on  $Z$ . Instead of  $\mathcal{X} \rightarrow \mathcal{Y}$ , we may put some class of operators; for example,  $A = \mathcal{O}_Z(f)_{\Psi^k}$  means that for any fixed value of  $Z$ , the operator  $f^{-1}A$  lies in  $\Psi^k$ . This is stronger than estimating some norms of the full symbol of  $A$ , as the classes  $\Psi^k$  are not preserved under multiplication by functions of  $h$  that are not polynomials.

One can also define wavefront sets for operators that are not pseudodifferential. Let  $X_1$  and  $X_2$  be two manifolds. A properly supported operator  $B : C^\infty(X_2) \rightarrow \mathcal{D}'(X_1)$  is called *polynomially bounded*, if for each  $\chi_j \in C_0^\infty(X_j)$  and each  $s$ , there exists  $N$  such that  $\chi_1 B \chi_2$  is  $\mathcal{O}(h^{-N})$  as an operator  $H_h^s \rightarrow H_h^{-N}$  and  $H_h^N \rightarrow H_h^s$ . A product of such  $B$  with an operator that is  $\mathcal{O}(h^\infty)_{\Psi^{-\infty}}$  will also be  $\mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ . For a polynomially bounded  $B : \mathcal{D}'(X_2) \rightarrow C_0^\infty(X_1)$ , its wavefront set  $\text{WF}_h(B) \subset \overline{T^*}X_1 \times \overline{T^*}X_2$  is defined as follows: a point  $(x_1, \xi_1, x_2, \xi_2)$  does not lie in  $\text{WF}_h(B)$ , if there exist neighborhoods  $U_j$  of  $(x_j, \xi_j)$  in  $\overline{T^*}X_j$  such that for each  $A_j \in \Psi^{k_j}(X_j)$  with  $\text{WF}_h(A_j) \subset U_j$ , we have  $A_1 B A_2 = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ .

If  $X_1 = X_2 = X$ , then we call a polynomially bounded operator  $B$  *pseudolocal* if its wavefront set is a subset of the diagonal; in this case we consider  $\text{WF}_h(B)$  as a subset of  $\overline{T^*}X$ . Pseudolocality is equivalent to saying that for any  $A_j \in \Psi^{k_j}(X)$  with  $\text{WF}_h(A_1) \cap \text{WF}_h(A_2) = \emptyset$ , we have  $A_1 B A_2 = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ . The operators in  $\Psi^k$  are pseudolocal and the definition of their wavefront set given here agrees with the one given earlier; however, the definition in this paragraph can also be applied to operators with exotic symbols of §5.1.

A polynomially bounded operator  $B$  is called *compactly microlocalized*, if its wavefront set is a compact subset of  $T^*X_1 \times T^*X_2$ , and in particular does not intersect the fiber infinities  $S^*X_1 \times \overline{T^*}X_2$  and  $\overline{T^*}X_1 \times S^*X_2$ . In this case, if  $A_j \in \Psi^{\text{comp}}(X_j)$  are equal to the identity microlocally near the projections of  $\text{WF}_h(B)$  onto  $\overline{T^*}X_j$ , then  $B = A_1 B A_2 + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ . Every compactly microlocalized operator  $B$  is smoothing; we say that  $B = \mathcal{O}(h^r)$  for some  $r$ , if for each  $\chi_j \in C_0^\infty(X_j)$ , there exist  $s, s'$  such that

$$\|\chi_1 B \chi_2\|_{H_h^s \rightarrow H_h^{s'}} = \mathcal{O}(h^r). \quad (3.2)$$

In fact, if  $B$  is compactly microlocalized and  $B = \mathcal{O}(h^r)$ , then (3.2) holds for all  $s, s'$ .

Finally, we say that  $A \in \Psi^k(X)$  is *elliptic* (as an element of  $\Psi^k$ ) on some set  $V \subset \overline{T^*X}$ , if  $\langle \xi \rangle^{-k} \sigma(A)$  does not vanish on  $V$ . If  $A$  is elliptic on the wavefront set of some  $B \in \Psi^{k'}(X)$ , then we can find an operator  $W \in \Psi^{k'-k}(X)$  such that  $B = WA + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ . This implies the following *elliptic estimate*, formulated here for the case of a compact  $X$ :

$$\|Bu\|_{H_h^{s+k-k'}(X)} \leq C\|Au\|_{H_h^s(X)} + \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}(X)}, \quad (3.3)$$

for all  $s, N$  and each  $u \in \mathcal{D}'(X)$  such that  $Au \in H_h^s$ .

Still assuming  $X$  compact, the *non-sharp Gårding inequality* says that if  $A \in \Psi^k(X)$ ,  $\Psi_1 \in \Psi^0(X)$ ,  $\langle \xi \rangle^{-k} \operatorname{Re} \sigma(A) > 0$  on  $\operatorname{WF}_h(\Psi_1)$ , then

$$\operatorname{Re} \langle A\Psi_1 u, \Psi_1 u \rangle \geq C^{-1} \|\Psi_1 u\|_{H_h^{k/2}(X)} - \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}(X)}. \quad (3.4)$$

See Lemma 5.3 for a proof of the corresponding statement in the  $\Psi_{1/2}^{\operatorname{comp}}$  calculus; the same proof works here. Of course if  $A$  is symmetric, we may drop  $\operatorname{Re}$  from (3.4).

With  $X$  still compact, the *sharp Gårding inequality* (see e.g. [Zw, Theorem 9.11]) says that if instead  $A \in \Psi^k(X)$ ,  $\Psi_1 \in \Psi^0(X)$ ,  $\langle \xi \rangle^{-k} \operatorname{Re} \sigma(A) \geq 0$  near  $\operatorname{WF}_h(\Psi_1)$ , then

$$\operatorname{Re} \langle A\Psi_1 u, \Psi_1 u \rangle \geq -Ch \|\Psi_1 u\|_{H_h^{(k-1)/2}(X)} - \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}(X)}. \quad (3.5)$$

**3.2. Quantization of canonical transformations.** In this section, we discuss local quantization of symplectic transformations. The resulting semiclassical Fourier integral operators will be needed to approximate the operator  $A$  from Lemma 2.2 by finite rank operators; see Lemma 2.2(3) and §6.1.

The theory described below can be found in [Al], [GuSt90, Chapter 6], [GuSt10, Chapter 8], [VũNg, §2.3], or [Zw, Chapters 10–11]. For the closely related microlocal setting, see [HöIV, Chapter 25] or [GrSj, Chapters 10–11]. We follow in part [Dy12a, §2.3]. Note that we will only need the relatively simple, local part of the theory of Fourier integral operators, as we quantize canonical transformations locally and we do not use geometric invariance of the principal symbol. For a more complete discussion, see for example [DyGu, §3].

Let  $X_1, X_2$  be two manifolds of same dimension,  $U_j \subset T^*X_j$  two bounded open sets, and  $\varkappa : U_1 \rightarrow U_2$  a symplectomorphism. First, assume that

- $x_j$  are systems of coordinates on the projections  $\pi_j(U_j)$  of  $U_j$  onto  $X_j$ ;
- $(x_j, \xi_j)$  are the corresponding coordinates on  $U_j$ ;
- there exists a generating function  $S(x_1, \xi_2) \in C^\infty(U_S)$  for some open  $U_S \subset \mathbb{R}^{2n}$  such that in coordinates  $(x_1, \xi_1, x_2, \xi_2)$ , the graph of  $\varkappa$  is given by

$$x_2 = \partial_{\xi_2} S(x_1, \xi_2), \quad \xi_1 = \partial_{x_1} S(x_1, \xi_2).$$

Such coordinate systems and generating functions exist locally near every point in the graph of  $\varkappa$ , see for example the paragraph before the final remark of [GrSj, Chapter 9]. (The authors of [GrSj] consider the homogeneous case, but this does not make a difference



here except possibly at the points of the zero section of  $T^*X_2$ . At these points, a different parametrization is possible and all the results below still hold, but we do not present this parametrization since the canonical transformations we use can be chosen to avoid the zero section.) An operator  $B : C^\infty(X_2) \rightarrow C^\infty(X_1)$  of the form

$$(Bf)(x_1) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{i(S(x_1, \xi_2) - x_2 \cdot \xi_2)/h} b(x_1, \xi_2, x_2; h) f(x_2) dx_2 d\xi_2, \quad (3.6)$$

where  $b \in C_0^\infty(U_S \times \pi_2(U_2))$  is a classical symbol in  $h$  (namely, it is a smooth function of  $h \geq 0$  up to  $h = 0$ ), is called a local Fourier integral operator associated to  $\varkappa$ . Such an operator is polynomially bounded and compactly microlocalized, in the sense of §3.1.

In general, we call  $B : C^\infty(X_2) \rightarrow C^\infty(X_1)$  a (compactly microlocalized) Fourier integral operator associated to  $\varkappa$ , if it can be represented as a finite sum of expressions of the form (3.6) with various choices of local coordinate systems (and thus various generating functions) plus an  $\mathcal{O}(h^\infty)_{\Psi^{-\infty}}$  remainder. Note that we use the convention that  $B$  acts  $C^\infty(X_2) \rightarrow C^\infty(X_1)$ , which is opposite to the more standard convention that  $B$  acts  $C^\infty(X_1) \rightarrow C^\infty(X_2)$ ; in the latter convention, we would say that  $B$  quantizes  $\varkappa^{-1}$ .

Here are some properties of (compactly microlocalized) Fourier integral operators:

- (1) if  $B$  is a Fourier integral operator associated to  $\varkappa$ , then  $\text{WF}_h(B)$  is a compact subset of the graph of  $\varkappa$ ;
- (2) if  $B$  is associated to  $\varkappa$ , then the adjoint  $B^*$  is associated to  $\varkappa^{-1}$ ;
- (3) if  $X_1 = X_2 = X$ , and  $\varkappa$  is the identity map on some open bounded  $U \subset T^*X$ , then  $B$  is associated to  $\varkappa$  if and only if  $B \in \Psi^{\text{comp}}(X)$  and  $\text{WF}_h(B) \subset U$ ;
- (4) if  $B$  is associated to  $\varkappa$  and  $B'$  is associated to  $\varkappa'$ , then  $BB'$  is associated to  $\varkappa' \circ \varkappa$ ;
- (5) if  $B$  is associated to  $\varkappa$ , then it has norm  $\mathcal{O}(1)$  in the sense of §3.1. This property follows from the previous three, as  $B^*B \in \Psi^{\text{comp}}(X_2)$ .
- (6) (Egorov's theorem) If  $A_j \in \Psi^{k_j}(X_j)$  and  $\sigma(A_1) = \sigma(A_2) \circ \varkappa$  near the projection of  $\text{WF}_h(B)$  onto  $T^*X_1$ , then  $A_1B = BA_2 + \mathcal{O}(h)$ . Here  $\mathcal{O}(h)$  is understood in the sense of (3.2), as both sides of the equation are compactly microlocalized.

If  $K_j \subset U_j$  are compact sets such that  $\varkappa(K_1) = K_2$ , then we say that a pair of operators

$$B : C^\infty(X_2) \rightarrow C^\infty(X_1), \quad B' : C^\infty(X_1) \rightarrow C^\infty(X_2)$$

quantizes  $\varkappa$  near  $K_1 \times K_2$ , if  $B, B'$  are compactly microlocalized Fourier integral operators associated to  $\varkappa, \varkappa^{-1}$ , respectively, and

$$BB' = 1, \quad B'B = 1$$

microlocally near  $K_1$  and  $K_2$ , respectively.

Such a pair  $(B, B')$  can be found for any given  $\varkappa$ , if we shrink  $U_j$  sufficiently, as given by the following construction of [Zw, Chapter 11]. First of all, we pass to local coordinates to



assume that  $X_j = \mathbb{R}^n$ . Next, by [Zw, Theorem 11.4] (putting  $\kappa = \varkappa^{-1}$ ,  $\tilde{\kappa}_t = \varkappa_t^{-1}$ ,  $q_t = -z_t$ ), we can construct a smooth family of symplectomorphisms  $\varkappa_t, t \in [0, 1]$  on  $T^*\mathbb{R}^n$  such that

- $\varkappa_t$  is equal to the identity outside of some fixed compact set;
- $\varkappa_0 = 1_{T^*\mathbb{R}^n}$  and  $\varkappa_1$  extends  $\varkappa$ ;
- there exists a family of real-valued functions  $z_t \in C_0^\infty(T^*\mathbb{R}^n)$  such that for each  $t \in [0, 1]$  and  $H_{z_t}$  the Hamiltonian vector field of  $z_t$ ,

$$(\partial_t \varkappa_t) \circ \varkappa_t^{-1} = H_{z_t}. \quad (3.7)$$

In other words,  $\varkappa$  is a deformation of the identity along the Hamiltonian flow of the time-dependent function  $z_t$ .

Let  $Z_t$  be the Weyl quantization of  $z_t$ ; this is a self-adjoint operator on  $L^2(\mathbb{R}^n)$ . Consider the family of unitary operators  $B_t$  on  $L^2(\mathbb{R}^n)$  solving the equations [Zw, Theorem 10.1]

$$hD_t B_t = B_t Z_t, \quad B_0 = 1. \quad (3.8)$$

By [Zw, Theorem 10.3] (and using the composition property (4) above to pass from small  $t$  to  $t = 1$ ) we see that, if  $\Psi_j \in \Psi^{\text{comp}}(X_j)$  with  $\text{WF}_h(\Psi_j) \subset U_j$  and  $\Psi_j = 1$  microlocally near  $K_j$ , then  $(B, B') = (\Psi_1 B_1 \Psi_2, \Psi_2 B_1^{-1} \Psi_1)$  quantizes  $\varkappa$  near  $K_1 \times K_2$ .

The difference between (3.8) and [Zw, (10.2.1)] is explained by the fact that we quantize a transformation  $T^*X_1 \rightarrow T^*X_2$  as an operator  $C^\infty(X_2) \rightarrow C^\infty(X_1)$ , while [Zw] quantizes it as an operator  $C^\infty(X_1) \rightarrow C^\infty(X_2)$ ; in the latter convention,  $U(t) = B_t^*$  and  $P(t) = -Z_t$ .

#### 4. ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

**4.1. Review of the construction of [Va11].** Throughout this section,  $|\text{Re } z| \leq C_0 h$  and  $\text{Im } z \geq -C_0 h$ . Our notation will differ in several places from the one used in [Va11]. In particular, we use  $(x, \xi)$  to denote coordinates on the whole  $T^*X$ ,  $(\tilde{x}, \tilde{y})$  for the product coordinates near the conformal boundary of  $M$ ,  $\tilde{\xi}$  for the momentum corresponding to  $\mu = \tilde{x}^2$ , and  $\tilde{\eta}$  for the momentum corresponding to  $\tilde{y}$ .

Let  $(M, g)$  be an even asymptotically hyperbolic manifold; we consider  $\delta_0 > 0$  and a boundary defining function  $\tilde{x}$  on  $\overline{M}$  such that  $\{\tilde{x} < \delta_0^2\} \simeq [0, \delta_0^2] \times \partial \overline{M}$  and the metric  $g$  has the form (1.2) for some metric  $g_1$  depending smoothly on  $\tilde{x}^2$ . Consider the space  $\overline{M}_{\text{even}}$ , which is topologically  $\overline{M}$ , but with smooth structure at the boundary changed so that

$$\mu = \tilde{x}^2$$

is a boundary defining function. Now, we consider the modified Laplacian

$$P_1(z) = \mu^{-\frac{1}{2} - \frac{n+1}{4}} e^{\frac{i(z+1)\phi}{h}} (h^2(\Delta_g - (n-1)^2/4) - (z+1)^2) e^{-\frac{i(z+1)\phi}{h}} \mu^{-\frac{1}{2} + \frac{n+1}{4}}. \quad (4.1)$$

Here  $\phi$  is a smooth real-valued function on  $M$  such that

$$e^\phi = \mu^{1/2}(1 + \mu)^{-1/4} \text{ on } \{0 < \mu < \delta_0\}.$$

The function  $\phi$  satisfies additional assumptions given in the proof of Lemma 4.3. This lemma, needed for the proof of the improved estimate in the upper half-plane (2.3), is the only place where the factor  $(1 + \mu)^{-1/4}$  is needed; the rest of the analysis would work if simply  $e^\phi = \mu^{1/2}$  on all of  $M$ .

As computed in [Va11, (3.5)],  $P_1(z)$  has coefficients smooth up to the boundary of  $\overline{M}_{\text{even}}$ . For  $\delta_0 > 0$  small enough, this operator continues smoothly to

$$X_{-\delta_0} = \{\mu > -\delta_0\},$$

by which we mean  $\overline{M}_{\text{even}} \cup \{|\mu| < \delta_0\}$ , where  $\{|\mu| < \delta_0\}$  is the double space of  $\{0 \leq \mu < \delta_0\}$ . (See [Va11, §3.1] for more details.)

**Lemma 4.1.** *There exists a manifold  $X$  without boundary and a family of operators  $P(z)$  so that:*

- (1)  $P(z) \in \Psi^2(X)$  depends holomorphically on  $z$  and  $p = \sigma(P(0))$  is real valued;
- (2)  $X_{-\delta_0}$  embeds into  $X$  and  $\mu$  continues to  $X$  so that  $X_{-\delta_0} = \{\mu > -\delta_0\}$ ;
- (3) the restriction of  $P(z)$  to  $X_{-\delta_0}$  is equal to  $P_1(z)$ ;
- (4) the characteristic set  $\{\langle \xi \rangle^{-2} p = 0\}$  is the disjoint union of two closed sets  $\Sigma_+$  and  $\Sigma_-$ , with  $\Sigma_\pm = \{\langle \xi \rangle^{-2} p = 0\} \cap \{\pm \tilde{\xi} > 0\}$  near  $\{\mu = 0\} \cap S^*X$ ;
- (5)  $\Sigma_+ \cap S^*X \subset \{\mu \leq 0\}$  and  $\Sigma_- \subset \{\mu \leq 0\}$ ;
- (6)  $P(z) - P(0) = (hL + a_0 + a_1 h)z + a_2 z^2$ , with the vector field  $L$  and the functions  $a_0, a_1, a_2$  independent of  $z$  and  $h$ .

*Proof.* The fact that the characteristic set of  $p$  on  $X_{-\delta_0}$  is the disjoint union of two sets  $\Sigma'_+$  and  $\Sigma'_-$ , with  $\Sigma'_\pm$  satisfying (4) and (5), is proven in [Va11, §3.4], with  $\Sigma_\pm$  denoted by  $\Sigma_{h,\pm}$  there. The manifold  $X$  is taken to be the double space of  $X_{-\delta_0}$ ; the extension of  $P_1(z)$  and  $\Sigma'_\pm$  to  $X_{-\delta_0}$  is constructed in [Va11, §3.5]. The formula (6) follows from (4.1).  $\square$

Consider the product coordinates  $(\mu, \tilde{y})$  near  $\{\mu = 0\}$ , the corresponding momenta  $(\tilde{\xi}, \tilde{\eta})$ , and define

$$L_\pm = \{\mu = 0, \tilde{\xi} = \pm\infty, \tilde{\eta} = 0\} \subset \Sigma_\pm \cap S^*X; \quad (4.2)$$

they can be viewed as images of the conical sets  $\{\mu = 0, \pm \tilde{\xi} > 0, \tilde{\eta} = 0\}$  in  $S^*M$ .

**Lemma 4.2.** *The ‘event horizon’  $\{\mu = 0\}$  has the following properties:*

- (1)  $L_\pm$  consists of fixed points for  $\langle \xi \rangle^{-1} H_p$  (also called radial points), with  $L_+$  a source and  $L_-$  a sink;
- (2)  $\Sigma_+ \cap S^*X \cap \{\mu = 0\} = L_+$  and  $\Sigma_- \cap \{\mu = 0\} = L_-$ ;
- (3) for  $\delta > 0$  small enough,  $\pm \langle \xi \rangle^{-1} H_p \mu < 0$  on  $\Sigma_\pm \cap \{-\delta \leq \mu \leq 0\} \setminus L_\pm$ ;

From this Lemma it follows that bicharacteristics of  $H_p$  can cross  $\{\mu = 0\}$  only from  $\{\mu > 0\}$  to  $\{\mu < 0\}$  and never in the other direction, hence the name ‘event horizon’ as in the theory of black holes [Va10].

*Proof.* (1) This is proved in [Va11, §3.4]; see (4.11) below for a quantification of the source/sink property.

(2) Near  $\{\mu = 0\}$ , we have [Va11, (3.23)]

$$p = 4\mu\tilde{\xi}^2 - 4(1 + \mathcal{O}(\mu))\tilde{\xi} - 1 + \mathcal{O}(\mu) + g_1^{-1}(\tilde{\eta}, \tilde{\eta}). \quad (4.3)$$

Here  $\mathcal{O}(\mu)$  denotes a smooth function of  $\mu$  vanishing at  $\mu = 0$ . Take  $\check{\xi} = \langle \xi \rangle^{-1}\tilde{\xi}$ ,  $\check{\eta} = \langle \xi \rangle^{-1}\tilde{\eta}$ ; then at  $\{\mu = 0\}$ ,

$$\langle \xi \rangle^{-2}p = -4\langle \xi \rangle^{-1}\check{\xi} - \langle \xi \rangle^{-2} + g_1^{-1}(\check{\eta}, \check{\eta}),$$

This shows that  $\Sigma_{\pm} \cap \{\mu = 0\} \cap S^*X = L_{\pm}$ ; to see that  $\Sigma_{-} \cap \{\mu = 0\} \subset S^*X$ , we can use that  $\Sigma_{\pm} \cap \{\mu = 0\} \subset \{\pm(\tilde{\xi} + 1/2) > 0\}$ , as shown in the discussion following [Va11, (3.27)], together with (4.3).

(3) Take  $\check{\xi}, \check{\eta}$  as in part (2); using (4.3), we get

$$\langle \xi \rangle^{-1}H_p\mu = -4\langle \xi \rangle^{-1}(1 + \mathcal{O}(\mu)) + 8\mu\check{\xi}. \quad (4.4)$$

On  $\Sigma_{+}$ , as in part (2),  $\tilde{\xi} > -1/2$  and thus  $\check{\xi} \geq -\langle \xi \rangle^{-1}/2$ ; therefore, for  $\mu \leq 0$ ,

$$\langle \xi \rangle^{-1}H_p\mu \leq -4\langle \xi \rangle^{-1}(1 + \mathcal{O}(\mu)) - 4\mu\langle \xi \rangle^{-1}$$

This is negative for small  $\mu$  unless  $\langle \xi \rangle^{-1} = 0$ ; in the latter case,  $\langle \xi \rangle^{-1}H_p\mu = 8\mu\check{\xi}$  and  $\check{\xi} > 0$ ,  $\mu < 0$  since we are on  $\Sigma_{+} \cap S^*X \setminus L_{+}$ . On  $\Sigma_{-} \setminus S^*M$  we use (4.3) to eliminate the last term from (4.4), getting

$$\langle \xi \rangle^{-1}H_p\mu = 2\langle \xi \rangle^{-1}(2 + \tilde{\xi}^{-1})(1 + \mathcal{O}(\mu)) - 2g_1^{-1}(\tilde{\eta}, \tilde{\eta})\tilde{\xi}^{-1} \geq 2\langle \xi \rangle^{-1}(2 + \tilde{\xi}^{-1})(1 + \mathcal{O}(\mu)).$$

Since  $\tilde{\xi} < -1/2$  on  $\Sigma_{-}$ , and since  $\tilde{\xi}$  must attain a maximum there, this expression is positive for small  $\mu$ . Finally, on  $\Sigma_{-} \cap S^*M \setminus L_{-}$  we have  $\langle \xi \rangle^{-1}H_p\mu = 8\mu\check{\xi}$  and  $\mu < 0$ ,  $\check{\xi} < 0$ .  $\square$

We also need to compute the sign of the imaginary part of  $P(z)$  when  $z$  moves away from the real line. We will use this to obtain improved estimates (2.3) in the physical half-plane.

**Lemma 4.3.** *The operator  $\partial_z P(z)$  lies in  $\Psi^1(X)$  and for an appropriate choice of the function  $\phi$  from (4.1) and  $\delta > 0$  small enough, we have*

$$\mp \operatorname{Re}(\langle \xi \rangle^{-1}\sigma(\partial_z P(0))) > 0 \text{ near } \Sigma_{\pm} \cap \{\mu \geq -\delta\}. \quad (4.5)$$

*Proof.* It follows from part (6) of Lemma 4.1 that  $\partial_z P(z) \in \Psi^1(X)$ . Moreover, the principal symbol of  $\partial_z P(z)$  is equal to the derivative in  $z$  of the principal symbol of  $P(z)$ .

For  $\delta$  small enough, we can use [Va11, (3.6)] to write the principal symbol of  $P(z)$  in  $\{|\mu| \leq \delta\}$  as

$$4\mu\tilde{\xi}^2 - 4(1 + \mathcal{O}(\mu))(1 + z)\tilde{\xi} - (1 + z)^2(1 + \mathcal{O}(\mu)) + g_1^{-1}(\tilde{\eta}, \tilde{\eta}).$$

Therefore, the principal symbol of  $\partial_z P(z)$  at  $z = 0$  is

$$-4(1 + \mathcal{O}(\mu))\tilde{\xi} - 2 + \mathcal{O}(\mu).$$

Since  $\Sigma_{\pm} \cap \{|\mu| \leq \delta\} \subset \{\pm(\tilde{\xi} + 1/2) > 0\}$ , we obtain (4.5) for  $|\mu| \leq \delta$ .

It remains to consider the region  $\{\mu > \delta\}$ . Choose the function  $\phi$  so that  $|d\phi|_{g^{-1}} < 1$ , this is possible by the discussion following [Va11, (3.13)]. By [Va11, (3.11)], we have

$$\sigma(\partial_z P(0)) = -2\mu^{-1}((\xi, d\phi)_{g^{-1}} + 1 - |d\phi|_{g^{-1}}^2).$$

Since  $(\xi - d\phi, \xi - d\phi)_{g^{-1}} = 1$  on  $\Sigma_+$ , this becomes

$$-\mu^{-1}(|\xi|_{g^{-1}}^2 + 1 - |d\phi|_{g^{-1}}^2) < 0. \quad \square$$

To relate  $p$  to the principal symbol  $p_0$  of  $h^2\Delta_g - 1$ , consider the map  $\iota : T^*M \rightarrow \overline{T^*}X$  given by

$$\iota(x, \xi) = (x, \xi + d\phi(x)), \quad x \in M, \quad \xi \in T_x^*M; \quad (4.6)$$

then

$$p(\iota(x, \xi)) = \mu(x)^{-1}p_0(x, \xi). \quad (4.7)$$

In particular, the images of flow lines of  $H_{p_0}$  on  $p_0^{-1}(0)$  under  $\iota$  are reparametrized flow lines of  $H_p$  on  $p^{-1}(0)$ ; the reparametrization factor is bounded as long as we are away from  $\{\mu \leq 0\}$ .

To state our next lemma, which collects some global properties of the flow of  $H_p$ , we need the following notions of sets trapped in one time direction on  $T^*M$ :

$$\tilde{\Gamma}_\pm = \{\rho \in T^*M \setminus 0 \mid \{\exp(tH_{p_0})\rho \mid \mp t \geq 0\} \text{ is bounded}\}, \quad \Gamma_\pm = \tilde{\Gamma}_\pm \cap p_0^{-1}(0). \quad (4.8)$$

The sets  $\tilde{\Gamma}_+$  and  $\tilde{\Gamma}_-$  are respectively the *forward* and *backward trapped sets*, and  $\tilde{K} = \tilde{\Gamma}_+ \cap \tilde{\Gamma}_-$ ,  $K = \Gamma_+ \cap \Gamma_-$ .

**Lemma 4.4.** *If  $\delta > 0$  is small enough and  $\gamma(t)$  is a flow line of  $\langle \xi \rangle^{-1}H_p$  on  $\{\langle \xi \rangle^{-2}p = 0\}$  with  $\gamma(0) \in \{\mu > -\delta\}$ , then:*

- (1) *if  $\gamma(0) \in \Sigma_+ \setminus (L_+ \cup \iota(\Gamma_-))$ , then there exists  $T > 0$  such that  $\gamma(T) \in \{\mu \leq -\delta\}$ ;*
- (2) *if  $\gamma(0) \in \Sigma_+$ , then either  $\gamma(-T) \in \iota(\Gamma_+)$  for  $T > 0$  large enough or  $\gamma(t)$  converges to  $L_+$  as  $t \rightarrow -\infty$ ;*
- (3) *if  $\gamma(0) \in \Sigma_- \setminus L_-$ , then there exists  $T > 0$  such that  $\gamma(-T) \in \{\mu \leq -\delta\}$ ;*
- (4) *if  $\gamma(0) \in \Sigma_-$ , then  $\gamma(t)$  converges to  $L_-$  as  $t \rightarrow +\infty$ .*

See Figure 2 in the introduction for a picture of the global dynamics of the flow.

*Proof.* We demonstrate (1); the other statements are proved similarly. By [Va11, Lemma 3.2], there exists  $T_0 \geq 0$  such that  $\gamma(T_0) \in \{|\mu| \geq \delta\}$ . If  $\mu(\gamma(T_0)) \leq -\delta$ , then we are done; assume that  $\mu(\gamma(T_0)) \geq \delta$ . By [Va11, (3.30)] and Lemma 4.2(3), the set  $V_\delta = \Sigma_+ \cap \{\mu \geq \delta\}$  is convex in the following sense: if  $\tilde{\gamma}(t)$  is any flow line of  $\langle \xi \rangle^{-1}H_p$  with  $\tilde{\gamma}(t_1), \tilde{\gamma}(t_2) \in V_\delta$  for some  $t_1 < t_2$ , then the whole segment  $\tilde{\gamma}([t_1, t_2])$  lies in  $V_\delta$ . This leaves only two cases: either  $\gamma([T_0, \infty)) \subset V_\delta$  or there exists  $T_1 > T_0$  such that  $\gamma([T_1, \infty)) \cap V_\delta = \emptyset$ . By (4.7), the first case would mean that  $\gamma(0) \in \iota(\Gamma_-)$ ; the second case implies by [Va11, Lemma 3.2] that  $\gamma(T) \in \{\mu = -\delta\}$  for some  $T > 0$ .  $\square$

Now, we take  $\delta$  small enough so that Lemmas 4.2, 4.3, and 4.4 hold and any operator  $Q$  such that:

- $Q \in \Psi^2(X)$  with Schwartz kernel supported in  $\{\mu < -\epsilon\}$  for some  $\epsilon > 0$ ;
- $q = \sigma(Q)$  is real-valued and  $\pm q \geq 0$  near  $\Sigma_\pm$ ;
- $Q$  is elliptic on  $\{\langle \xi \rangle^{-2} p = 0\} \cap \{\mu \leq -\delta\}$ .

Such a  $Q$  satisfies the conditions of [Va11, §3.5], except for the self-adjointness condition; Lemma 2.1 then follows from [Va11, Theorem 4.3] and [Va11, proof of Theorem 5.1]. (In [Va11],  $Q$  was required to be self-adjoint; however as remarked in [Va10, §2.2], this condition can be relaxed. Strictly speaking we are citing [Va10, Theorem 2.11, Theorem 4.3]). We will impose more conditions on  $Q$  in §7.

**4.2. Conjugation and escape function.** We first study  $P(z)$  near the radial points  $L_\pm$  defined in (4.2). The functions  $\mu, \tilde{y}, \tilde{\rho} = |\tilde{\xi}|^{-1}, \hat{\eta} = \tilde{\rho}\tilde{\eta}$  form a coordinate system on  $\overline{T^*X}$  near  $L_\pm$ ; in these coordinates,  $L_\pm$  is given by  $\{\mu = \tilde{\rho} = \hat{\eta} = 0\}$ , and [Va11, (3.23) and (3.28)]

$$\begin{aligned} |\tilde{\xi}|^{-2} p &= 4\mu \mp 4(1 + \mathcal{O}(\mu))\tilde{\rho} - (1 + \mathcal{O}(\mu))\tilde{\rho}^2 + g_1^{-1}(\hat{\eta}, \hat{\eta}); \\ |\tilde{\xi}|^{-1} H_p &= \pm 4(2\mu\partial_\mu + \tilde{\rho}\partial_{\tilde{\rho}} + \hat{\eta}\partial_{\hat{\eta}}) - 4\tilde{\rho}\partial_\mu + \mathcal{O}(\hat{\eta})\partial_{\tilde{y}} + \mathcal{O}(\mu^2 + \tilde{\rho}^2 + |\hat{\eta}|^2). \end{aligned} \quad (4.9)$$

Define the function

$$\rho_1 = \mu^2 + \tilde{\rho}^2 + g_1^{-1}(\hat{\eta}, \hat{\eta}) \quad (4.10)$$

near  $L_\pm$ ; extend it to the whole  $\overline{T^*X}$  so that  $\rho_1 > 0$  outside of  $L_+ \cup L_-$ . The function  $\rho_1$  has properties similar to the function  $\tilde{\rho}^2 + \rho_0$  used in the proof of [Va11, Proposition 4.6]; we will use it to define neighborhoods of  $L_\pm$ .

**Lemma 4.5.** *For  $\delta > 0$  small enough,*

$$\pm \langle \xi \rangle^{-1} H_p \rho_1 \geq \delta \rho_1 \text{ on } \Sigma_\pm \cap \{\rho_1 \leq 5\delta\}; \quad (4.11)$$

$$\{\langle \xi \rangle^{-2} p = 0\} \cap S^*X \cap \{\mu \geq -\delta\} \subset \{\rho_1 < 5\delta\}. \quad (4.12)$$

*Proof.* We compute  $\pm |\tilde{\xi}|^{-1} H_p \rho_1 \geq 4\rho_1 - \mathcal{O}(\rho_1^{3/2})$ ; (4.11) follows. To show (4.12), take a point in  $\Sigma_\pm \cap S^*X \cap \{\mu \geq -\delta\}$ ; by (4.7), we have  $\mu \leq 0$ . By Lemma 4.2(2), for  $\delta$  small enough our point lies in the domain of the coordinate system  $(\mu, \tilde{y}, \tilde{\rho}, \hat{\eta})$ . By (4.9), and using  $S^*X = \{\tilde{\rho} = 0\}$ , we get

$$g_1^{-1}(\hat{\eta}, \hat{\eta}) = -4\mu \leq 4\delta;$$

it remains to recall the definition of  $\rho_1$ . □

We now take the density on  $X$  introduced in [Va11, §3.1] (in fact, any density would work). If  $B$  is any continuous operator  $C^\infty(X) \rightarrow \mathcal{D}'(X)$  and  $B^*$  is its adjoint, then define

$$\text{Im } B = \frac{1}{2i}(B - B^*);$$

for each  $u \in C^\infty(X)$ ,

$$\operatorname{Im}(Bu, u) = ((\operatorname{Im} B)u, u).$$

Instead of using the radial points estimate [Va11, Proposition 4.5], we will conjugate  $P(z)$  by an elliptic operator of order  $s$  to make the imaginary part of the subprincipal symbol have the correct sign:

**Lemma 4.6.** *Assume that  $|\operatorname{Re} z| \leq C_0 h$ ,  $|\operatorname{Im} z| \leq C_0 h$ ,  $s > C_0$  is fixed, and  $T_s \in \Psi^s(X)$  is any elliptic operator, as in Lemma 2.2. Then for  $\delta > 0$  small enough,*

$$\mp \sigma(h^{-1} \operatorname{Im}(T_s P(z) T_s^{-1})) \geq \delta \langle \xi \rangle \text{ on } \Sigma_\pm \cap \{\rho_1 \leq 5\delta\}.$$

Note that, since  $\sigma(T_s P(z) T_s^{-1}) = p$  is real-valued,  $\operatorname{Im}(T_s P(z) T_s^{-1}) \in h\Psi^1$ .

*Proof.* We consider  $P(z)$  as a function of  $\tilde{z} = h^{-1}z$ . It suffices to show that

$$\pm \langle \xi \rangle^{-1} \sigma(h^{-1} \operatorname{Im}(T_s P(z) T_s^{-1}))|_{L_\pm} < 0.$$

Consider the coordinates  $\mu, \tilde{y}, \tilde{\rho} = |\tilde{\xi}|^{-1}, \hat{\eta} = \tilde{\rho} \tilde{\eta}$  near  $L_\pm$ . By [Va11, (3.10)],

$$|\tilde{\xi}|^{-1} \sigma(h^{-1} \operatorname{Im} P(z))|_{L_\pm} = \mp 4 \operatorname{Im} \tilde{z}.$$

Furthermore, by part (6) of Lemma 4.1 we have  $P(z) - P(0) \in h\Psi^1$ , and hence  $T_s(P(z) - P(0))T_s^{-1} - (P(z) - P(0)) \in h^2\Psi^0$ , from which we conclude that it suffices to prove that

$$|\tilde{\xi}|^{-1} \sigma(h^{-1}(T_s P(0) T_s^{-1} - P(0)))|_{L_\pm} = \mp 4is. \quad (4.13)$$

Near  $L_\pm$ , we can write  $\sigma(T_s) = \tilde{\rho}^{-s} e^a$ , with  $a$  smooth on  $\overline{T^*X}$ . Then

$$\begin{aligned} |\tilde{\xi}|^{-1} \sigma(h^{-1}(T_s P(0) T_s^{-1} - P(0))) &= \tilde{\rho} \sigma(h^{-1}[T_s, P(0)] T_s^{-1}) \\ &= i \tilde{\rho}^{s+1} e^{-a} H_p(\tilde{\rho}^{-s} e^a) = -is H_p \tilde{\rho} + i \tilde{\rho} H_p a. \end{aligned}$$

The first term on the right-hand side gives (4.13) by (4.9), while the second one vanishes at  $L_\pm$  since  $|\tilde{\xi}|^{-1} H_p = 0$  there.  $\square$

Now, we construct an escape function for the region  $(\Sigma_+ \cup \Sigma_-) \cap \{\mu \geq -\delta\} \setminus (U_K \cup \{\rho_1 < 5\delta\})$ , where  $U_K$  is a neighborhood of  $\iota(K)$  in  $T^*X$ . This is based partly on [DaVa, Lemma 4.3] (see also [GéSj, Appendix], [VaZw, §4]).

**Lemma 4.7.** *For  $\delta > 0$  small enough and any sufficiently small neighborhood  $U_K$  of  $\iota(K)$ , there exists a smooth nonnegative function  $f_0$  on  $\overline{T^*X}$  such that:*

- (1)  $f_0$  is supported in  $\{\mu > -2\delta\}$  and away from  $S^*X$ ;
- (2)  $H_p f_0 \geq 0$  near  $(\Sigma_+ \cap \{\mu \geq -\delta\}) \cup \overline{U}_K$ , and  $H_p f_0 \leq 0$  near  $\Sigma_- \cap \{\mu \geq -\delta\}$ ;
- (3)  $\pm H_p f_0 > 0$  on  $V_\pm = \Sigma_\pm \cap \{\mu \geq -\delta\} \setminus (U_K \cup \{\rho_1 < 5\delta\})$ ;
- (4)  $H_p f_0 = 0$  near  $\overline{U}_K \cap \iota(\tilde{K})$ .

In fact, the function  $f_0$  we construct in the proof is identically 1 near  $\overline{U}_K \cap \iota(\tilde{K})$ .

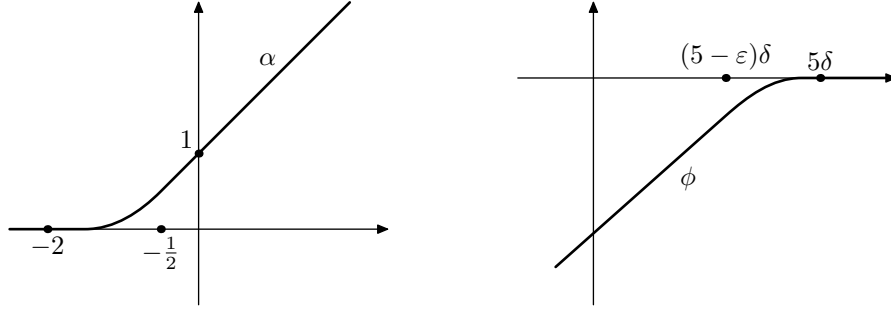


FIGURE 3. We precompose  $\tilde{f}_0$  with  $\alpha$  to obtain a function which is 0 near  $L_+$  and 1 near  $\tilde{K}$  and smooth in between. We use a multiple of  $\phi$  in the definition of  $\tilde{f}_0$  to guarantee that  $\tilde{f}_0 \leq -2$  near  $\Sigma_+ \cup \Sigma_- \cap S^*X$  so that  $\alpha \circ \tilde{f}_0$  vanishes there.

*Proof.* Note that  $V_+$  and  $V_-$  are compact and disjoint, and by (4.12) neither intersects  $S^*X$ . We will first construct a function  $\tilde{f}_0 \in C^\infty(\overline{T^*X})$ , with the following properties:

- (1)  $\tilde{f}_0 \leq -2$  near  $(\Sigma_+ \cup \Sigma_-) \cap S^*X \cap \{\mu \geq -\delta\}$ ;
- (2) near  $\Sigma_\pm \cap \{\mu \geq -\delta\}$ ,  $\pm H_p \tilde{f}_0 \geq 0$ ;
- (3)  $\pm H_p \tilde{f}_0 > 0$ ,  $\tilde{f}_0 \geq -1/2$  on  $V_\pm$ ;
- (4)  $H_p \tilde{f}_0 = 0$  near  $\iota(\tilde{K})$ .

Before constructing this function we show how we use it to construct  $f_0$ . In several places we will shrink  $U_K$ , keeping  $U_K \cap \Sigma_+$  fixed: note that this procedure does not affect  $\tilde{f}_0$ .

Take  $\alpha \in C^\infty(\mathbb{R})$  nondecreasing with  $\alpha(t) = t + 1$  near  $t \geq -1/2$  and  $\text{supp } \alpha \subset (-2, \infty)$ . (See Figure 3.) Take  $\chi \in C^\infty(\overline{T^*X}; [0, 1])$  supported in  $\{\mu > -2\delta\}$  and away from  $S^*X$ , and with  $\chi = 1$  near  $(\Sigma_+ \cup \Sigma_-) \cap \{\mu \geq -\delta\} \cap \{\tilde{f}_0 > -2\}$ . This is possible by property (1) of  $\tilde{f}_0$ . Then put

$$f_0(x, \xi) = \chi(x, \xi) \alpha(\tilde{f}_0(x, \xi)).$$

Property (1) of  $f_0$  follows from the support condition on  $\chi$ . Note that thanks to the choice of the set where  $\chi = 1$  together with  $\alpha(t) = 0$  near  $t \leq -2$ , we have

$$H_p f_0(x, \xi) = \alpha'(\tilde{f}_0(x, \xi)) H_p \tilde{f}_0 \text{ near } (\Sigma_+ \cup \Sigma_-) \cap \{\mu \geq -\delta\}. \quad (4.14)$$

Hence (if necessary shrinking  $U_K$  while keeping  $U_K \cap \Sigma_+$  fixed) property (2) of  $f_0$  follows from property (2) of  $\tilde{f}_0$  together with the fact that  $\alpha$  is nondecreasing. Properties (3) and (4) of  $f_0$  follow from properties (3) and (4) of  $\tilde{f}_0$  together with (4.14) and with the fact that  $\alpha(t) = t + 1$  near  $t \geq -1/2$ , again if necessary shrinking  $U_K$  while keeping  $U_K \cap \Sigma_+$  fixed.

We will take  $\tilde{f}_0$  of the form

$$\tilde{f}_0 = \sum_{k=1}^N f_k + C\phi(\rho_1).$$



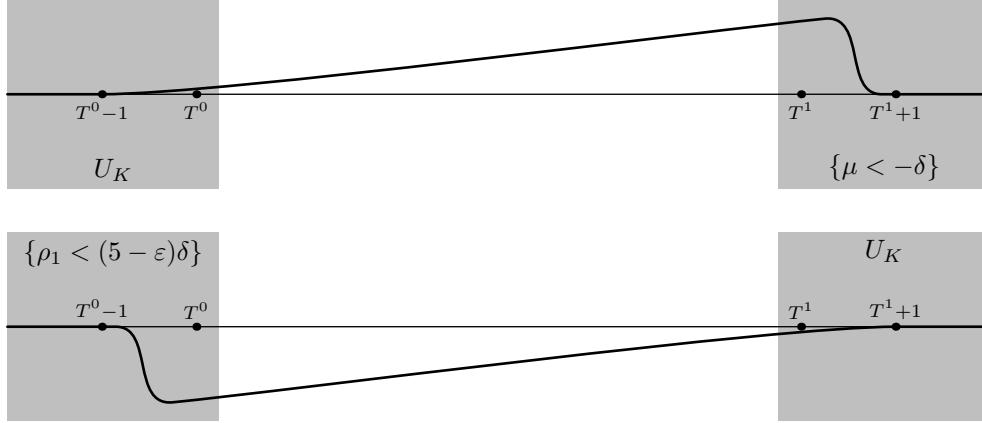


FIGURE 4. A graph of  $\chi_{(x,\xi)}(t)$  in the cases when the bicharacteristic through  $(x, \xi)$  tends to  $\iota(\tilde{K})$  as  $t \rightarrow -\infty$  (top) and as  $t \rightarrow +\infty$  (bottom) (other cases are similar and simpler). Here  $t$  is time along a bicharacteristic flowline. In each case  $\chi' \geq 0$  near the trapped set. In the first case  $\chi' > 0$  until the bicharacteristic enters the elliptic set of  $Q$ , and in the second case  $\chi' > 0$  starting when the bicharacteristic leaves a small neighborhood of the radial set and continuing until it enters a small neighborhood of the trapped set.

Here  $\phi \in C^\infty(\mathbb{R}; (-\infty, 0])$  is nondecreasing, supported in  $(-\infty, 5\delta]$ , and  $\phi' = 1$  on  $(-\infty, (5-\varepsilon)\delta]$ , where  $\varepsilon > 0$  is small enough that  $(\Sigma_+ \cup \Sigma_-) \cap S^*X \cap \{\mu \geq -\delta\} \subset \{\rho_1 < (5-\varepsilon)\delta\}$  (see (4.12)). Each  $f_k$ , specified below, is supported near the bicharacteristic through  $(x_k, \xi_k)$ , a suitably chosen point in  $V_+ \cup V_-$ . Now if  $C$  is sufficiently large (depending on  $\sum f_k$ ) we will have property (1) of  $\tilde{f}_0$ . It suffices now to construct the  $f_k$  so that properties (2), (3) and (4) of  $\tilde{f}_0$  hold, and indeed since  $\pm H_p \phi(\rho_1) \geq 0$  on  $\Sigma_\pm$  by (4.11) and  $\text{supp } \phi(\rho_1) \cap (V_+ \cup V_- \cup U_K) = \emptyset$  it is enough to check these properties for  $\sum f_k$  (to prove property (2) we will also increase  $C$  further).

To determine the  $(x_k, \xi_k)$  we first fix an open neighborhood  $\tilde{U}_K$  of  $\iota(K)$  with  $\overline{\tilde{U}_K} \subset U_K$ , and associate to each  $(x, \xi) \in V^+$  the following *escape times*:

$$\begin{aligned} T^0_{(x,\xi)} &= \inf\{t \in \mathbb{R} : \rho_1(\gamma(t)) \geq 4\delta \text{ and } \gamma(t) \notin \tilde{U}_K\}, \\ T^1_{(x,\xi)} &= \sup\{t \in \mathbb{R} : \mu(\gamma(t)) \geq -3\delta/2 \text{ and } \gamma(t) \notin \tilde{U}_K\}. \end{aligned}$$

Here  $\gamma(t)$  is the bicharacteristic flowline through  $(x, \xi)$ . For  $(x, \xi) \in V^-$  we put

$$T^0_{(x,\xi)} = \inf\{t \in \mathbb{R} : \mu(\gamma(t)) \geq -3\delta/2\}, \quad T^1_{(x,\xi)} = \inf\{t \in \mathbb{R} : \rho_1(\gamma(t)) \geq 4\delta\}.$$

Note that for every  $(x, \xi) \in V_- \cup V_+$  we have  $-\infty < T^0_{(x,\xi)} < 0 < T^1_{(x,\xi)} < \infty$  thanks to the description of the large time behavior of trajectories in  $\Sigma_\pm$  given by Lemma 4.4 (we use the fact that all trajectories in  $\iota(\Gamma_\pm)$  tend to  $\iota(K)$  as  $t \rightarrow \mp\infty$ , see for example [GéSj, Proposition A.2]). Next, let  $\mathcal{S}_{(x,\xi)}$  be a hypersurface through  $(x, \xi)$  which is transversal to

$H_p$  near  $(x, \xi)$ . Then if  $U_{(x, \xi)}$  is a sufficiently small neighborhood of  $(x, \xi)$ , the set

$$V_{(x, \xi)} = \{\gamma_{(x', \xi')}(t) : (x', \xi') \in U_{(x, \xi)} \cap \mathcal{S}_{(x, \xi)}, t \in (T_{(x, \xi)}^0 - 1, T_{(x, \xi)}^1 + 1)\},$$

where  $\gamma_{(x', \xi')}(t)$  is the bicharacteristic flowout of  $(x', \xi')$ , is diffeomorphic to  $(\mathcal{S}_{(x, \xi)} \cap U_{(x, \xi)}) \times (T_{(x, \xi)}^0 - 1, T_{(x, \xi)}^1 + 1)$ , and this diffeomorphism defines product coordinates on  $V_{(x, \xi)}$ . If necessary, shrink  $U_{(x, \xi)}$  so that

$$\overline{V_{(x, \xi)}} \cap \{t \in [T_{(x, \xi)}^0 - 1, T_{(x, \xi)}^0] \cup [T_{(x, \xi)}^1, T_{(x, \xi)}^1 + 1]\} \subset U_K \cup \{\rho_1 < (5 - \varepsilon)\delta\} \cup \{\mu < -\delta\}. \quad (4.15)$$

Take  $\varphi_{(x, \xi)} \in C_0^\infty(\mathcal{S}_{(x, \xi)} \cap U_{(x, \xi)}; [0, 1])$  identically 1 near  $(x, \xi)$ , also considered as a function on  $V_{(x, \xi)}$  via the product coordinates, and let  $V'_{(x, \xi)} \subset V_{(x, \xi)}$  be the product of  $(T_0, T_1)$  with an open subset of  $\mathcal{S}_{(x, \xi)} \cap U_{(x, \xi)}$  on which  $\varphi_{(x, \xi)} = 1$ . Using the compactness of  $V_+ \cup V_-$ , take  $(x_1, \xi_1), \dots, (x_N, \xi_N)$  with

$$V_+ \cup V_- \subset \bigcup_{k=1}^N V'_{(x_k, \xi_k)}.$$

For each  $k \in \{1, \dots, N\}$  put  $f_k = f_{(x_k, \xi_k)}$ , where

$$f_{(x, \xi)} = \chi_{(x, \xi)}(t)\varphi_{(x, \xi)}, \quad H_p f_{(x, \xi)} = \chi'_{(x, \xi)}(t)\varphi_{(x, \xi)},$$

and where  $\chi_{(x, \xi)} \in C_0^\infty((T_{(x, \xi)}^0 - 1, T_{(x, \xi)}^1 + 1))$ . Note that  $V_{(x, \xi)} \cap \iota(\tilde{K}) = \emptyset$  for all  $(x, \xi) \in V_+ \cup V_-$ , and so each  $f_k$  vanishes near  $\iota(\tilde{K})$ , and in particular we have property (4) of  $\tilde{f}_0$ .

We further impose that  $\pm\chi'_{(x, \xi)} > 0$  (accordingly as  $(x, \xi) \in V_\pm$ ) and  $\chi_{(x, \xi)} \geq -(2N)^{-1}$  on  $[T_{(x, \xi)}^0, T_{(x, \xi)}^1]$ . This condition gives property (3) of  $\tilde{f}_0$  since  $V_\pm \cap V_{(x, \xi)} \subset \{t \in [T_{(x, \xi)}^0, T_{(x, \xi)}^1]\}$  and  $\pm H_p f_{(x, \xi)} > 0$  on  $V'_{(x, \xi)}$ . If  $\gamma(t)$  (the bicharacteristic through  $(x, \xi)$ ) tends to  $\iota(\tilde{K})$  as  $t \rightarrow \pm\infty$ , then we further require that  $\chi'_{(x, \xi)}(t) \geq 0$  for  $\pm t \geq 0$ . (Note that  $\gamma(t)$  cannot tend to  $\iota(\tilde{K})$  both as  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$ , as in this case  $\gamma(t) \subset \iota(\tilde{K})$ .) This is sufficient to imply property (2) of  $\tilde{f}_0$  since now (4.15) implies that  $\Sigma_\pm \cap \{\mu \geq -\delta\} \cap \{\rho_1 \geq (5 - \varepsilon)\delta\} \cap V_{(x, \xi)} \subset \{\pm\chi'_{(x, \xi)}(t) \geq 0\}$ , and since the  $C\phi(\rho_1)$  term takes care of the set  $\{\rho_1 \leq (5 - \varepsilon)\delta\} \cap \Sigma_\pm$ .  $\square$

## 5. EXOTIC CLASSES OF OPERATORS

**5.1.  $\Psi_{1/2}$  calculus.** In this subsection we review the pseudodifferential calculus of operators with symbols in the exotic class  $S_{1/2}$  depending on two semiclassical parameters  $h, \tilde{h}$ , studied in [SjZw, §3.3] and [WuZw11, §3]. The escape function in §7.1 will provide positivity up to distance  $(h/\tilde{h})^{1/2}$  to the trapped set, and the operator  $A$  from Lemma 2.2 will be supported  $\mathcal{O}((h/\tilde{h})^{1/2})$  close to the trapped set; we will study both using this exotic class.

We always assume  $\tilde{h}$  is small but independent of  $h$ , and  $h$  is small depending on  $\tilde{h}$ . The reason for the second semiclassical parameter  $\tilde{h}$  and the corresponding symbol class  $S_{1/2}$  is the following: since our escape function is only regular on the scale  $h^{1/2}$  it belongs to a calculus with no asymptotic decomposition in powers of  $h$ , and some of the remainder

terms in the positive commutator estimate in §7 will be of order  $h$ , the same magnitude as the positive term coming from the commutator. However, if we use symbols in  $S_{1/2}$  (which are regular on the larger scale  $(h/\tilde{h})^{1/2}$  instead of just  $h^{1/2}$ ), then we have an asymptotic decomposition in powers of  $\tilde{h}$  for the corresponding calculus and the remainder terms will be  $\mathcal{O}(h\tilde{h})$ , and hence small in comparison with the  $h$  sized positive term for  $\tilde{h}$  small enough.

**Remark.** An alternative approach would use instead the mildly exotic class  $S_\rho$ , with  $\rho < 1/2$ ; its elements are regular on the scale  $h^\rho$ . The semiclassical calculus in this class has a decomposition in powers of  $h$ , which would simplify the arguments below, eliminating the need for  $\tilde{h}$ . However, in this case the rank of the operator  $A_R$  from Lemma 2.2 would grow as  $h^{-2\nu\rho-(n-1)(1-2\rho)}$ , which is the number of cylinders on the energy surface of size 1 in the direction of the Hamiltonian flow and size  $h^{1/2}$  in all other directions that are needed to cover an  $\mathcal{O}(h^\rho)$  neighborhood of the trapped set (see §6). This is a weaker estimate than the  $\mathcal{O}(h^{-\nu})$  that we prove. By taking  $\rho$  very close to  $1/2$ , one could get any power of  $h^{-1}$  bigger than  $\nu$ , but not  $h^{-\nu}$ ; this makes a difference if the trapped set is of pure Minkowski dimension, which is the case in the most interesting examples (see the introduction).

We proceed to the construction of the  $\Psi_{1/2}$  calculus. We will only need compactly microlocalized operators, thus we restrict ourselves to symbols that are  $\mathcal{O}(h^\infty)$  outside of a compact set. For a manifold  $X$ , we define the class  $S_{1/2}^{\text{comp}}(X)$  as follows: a function  $a(x, \xi; h, \tilde{h})$  smooth in  $(x, \xi) \in T^*X$  lies in this class if and only if:

- there exists a compact set  $V \subset T^*X$  such that each  $(x, \xi)$ -derivative of  $a$  is  $\mathcal{O}(h^\infty \langle \xi \rangle^{-\infty})$  outside of  $V$ , uniformly in  $\xi$  and locally uniformly in  $x$ ;
- for each multiindex  $\alpha$ , there exists a constant  $C_\alpha$  such that near  $V$ ,

$$|\partial_{x,\xi}^\alpha a| \leq C_\alpha (h/\tilde{h})^{-|\alpha|/2}. \quad (5.1)$$

As in §3.1, we require only local uniformity in  $x$ . This is in contrast with [SjZw] and [WuZw11], but their results still hold if we only require our estimates to be locally uniform in  $x$ .

We begin with operators on  $\mathbb{R}^n$ . For  $a \in S_{1/2}^{\text{comp}}(\mathbb{R}^n)$ , let  $\text{Op}_h(a)$  be its Weyl quantization:

$$\text{Op}_h(a)u(x) = (2\pi h)^{-n} \int e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) \tilde{\chi}(x-y)u(y) d\xi dy, \quad u \in C^\infty(\mathbb{R}^n). \quad (5.2)$$

Here  $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$  is some fixed function equal to 1 near the origin. We use the  $\tilde{\chi}(x-y)$  cutoff, which is absent in the standard definition of the Weyl quantization, to make  $\text{Op}_h(a)$  properly supported. It is also needed for the integral to converge, as  $a$  can grow arbitrarily fast as  $x \rightarrow \infty$ . The factor  $\tilde{\chi}(x-y)$  only changes the operator  $\text{Op}_h(a)$  by a smoothing term of order  $\mathcal{O}(h^\infty)$  because of the pseudolocality of  $\text{Op}_h(a)$ , see for example [WuZw11, Lemma 3.4]. (We will need to use more standard symbol classes (6.6) and the standard definition (6.7) of Weyl quantization in a limited way in §6.3.) Here are the basic properties of quantization of exotic symbols on  $\mathbb{R}^n$  (see [SjZw, §3.3] or [WuZw11, §3.2]):

**Lemma 5.1** (Properties of the  $S_{1/2}$  calculus on  $\mathbb{R}^n$ ).

1. For  $a \in S_{1/2}^{\text{comp}}(\mathbb{R}^n)$ ,  $\text{Op}_h(a)$  is compactly microlocalized, pseudolocal, and has norm  $\mathcal{O}(1)$ , in the sense of §3.1. If  $\text{supp } a \subset K$  for some compact set  $K \subset T^*\mathbb{R}^n$  independent of  $h, \tilde{h}$ , then  $\text{WF}_h(\text{Op}_h(a)) \subset K$ .

2. For  $a \in S_{1/2}^{\text{comp}}(\mathbb{R}^n)$ ,  $\text{Op}_h(a)^* = \text{Op}_h(\bar{a})$ .

3. For  $a, b \in S_{1/2}^{\text{comp}}(\mathbb{R}^n)$ , there exists a symbol  $a \# b \in S_{1/2}^{\text{comp}}(\mathbb{R}^n)$  such that

$$\text{Op}_h(a) \text{Op}_h(b) = \text{Op}_h(a \# b) + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

(The  $\mathcal{O}(h^\infty)_{\Psi^{-\infty}}$  error comes from the  $\tilde{\chi}(x-y)$  cutoff.) The same holds when one of  $a, b$  lies in  $S_{1/2}^{\text{comp}}(\mathbb{R}^n)$  and the other in the class  $S_{\text{cl}}^k(\mathbb{R}^n)$  defined in §3.1, with  $a \# b$  still in  $S_{1/2}^{\text{comp}}(\mathbb{R}^n)$ .

4. If  $a, b \in S_{1/2}^{\text{comp}}(\mathbb{R}^n)$ , then

$$a \# b = ab + \mathcal{O}(\tilde{h})_{S_{1/2}^{\text{comp}}(\mathbb{R}^n)}.$$

5. If one of  $a, b$  lies in  $S_{1/2}^{\text{comp}}(\mathbb{R}^n)$  and the other in  $S_{\text{cl}}^k(\mathbb{R}^n)$ , then

$$a \# b = ab + \mathcal{O}(h^{1/2} \tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}(\mathbb{R}^n)}, \quad a \# b - b \# a = -ih\{a, b\} + \mathcal{O}(h^{3/2} \tilde{h}^{3/2})_{S_{1/2}^{\text{comp}}(\mathbb{R}^n)}.$$

6. Assume that  $f : U_1 \rightarrow U_2$  is a diffeomorphism,  $U_1, U_2 \subset \mathbb{R}^n$ , and take  $\chi \in C_0^\infty(U_1)$ . Then for each  $a \in S_{1/2}^{\text{comp}}(\mathbb{R}^n)$ ,

$$(f^{-1})^* \chi \text{Op}_h(a) \chi f^* = \text{Op}_h(a_f) + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}, \quad a_f \in S_{1/2}^{\text{comp}}(\mathbb{R}^n),$$

$$a_f(x, \xi) = \chi(f^{-1}(x))^2 a(f^{-1}(x), {}^t f'(x)\xi) + \mathcal{O}(h^{1/2} \tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}.$$

This fact depends on using the Weyl quantization; the proof can be found in [WuZw11, Lemma 3.3]. (See also the proof of Lemma 5.5 below.)

Lemma 5.1(4) and (5) follow from the following asymptotic expansion [SjZw, Lemma 3.6]:

$$(a \# b)(x, \xi) \sim \sum_{j \geq 0} \frac{h^j}{j!(2i)^j} (\partial_\xi \cdot \partial_y - \partial_\eta \cdot \partial_x)^j a(x, \xi) b(y, \eta) \Big|_{\substack{y=x \\ \eta=\xi}}. \quad (5.3)$$

The expansion (5.3) holds in the following sense: for every  $N$ , each  $S_{1/2}$  seminorm of the difference of the left-hand side and the sum of the terms with  $j < N$  on the right-hand side is bounded by a certain  $S_{1/2}$  seminorm of the  $j = N$  term of the sum, taken without restricting to  $y = x, \eta = \xi$  (see [SjZw, (3.12)]). If both  $a, b$  are in  $S_{1/2}$ , then the  $j$ th term of the asymptotic sum is  $\mathcal{O}(\tilde{h}^j)$  and (5.3) is an expansion in powers of  $\tilde{h}$ , not  $h$ . However, if one of  $a, b$  lies in the class  $S_{\text{cl}}^k$ , then the  $j$ th term of the sum is  $\mathcal{O}(h^{j/2} \tilde{h}^{j/2})$  and we get an expansion in powers of  $h$  and better remainders for the product and commutator formulas. The improved remainder estimate for the commutator  $a \# b - b \# a$  in part 5 of Lemma 5.1 is due to the fact (specific to the Weyl quantization) that the  $j = 2$  term in (5.3) is the same for  $a \# b$  and  $b \# a$ ; therefore, the remainder comes from the  $j = 3$  term.

We can now construct the  $\Psi_{1/2}$  calculus on a manifold, similarly to [WuZw11, §3.3]. More specifically, we will define the class  $\Psi_{1/2}^{\text{comp}}(X)$  of compactly microlocalized operators with symbols in  $S_{1/2}^{\text{comp}}(X)$ . Let  $X$  be a manifold and  $A$  be a properly supported operator on  $X$  depending on  $h, \tilde{h}$ , which is compactly microlocalized and pseudolocal in the sense of §3.1. We say that  $A$  lies in  $\Psi_{1/2}^{\text{comp}}(X)$ , if for each coordinate system  $f : U_f \rightarrow V_f$ , with  $U_f \subset X$ ,  $V_f \subset \mathbb{R}^n$ , and each  $\chi \in C_0^\infty(U_f)$ , there exists  $a_{f,\chi} \in S_{1/2}^{\text{comp}}(U_f) \cap C_0^\infty(T^*U_f)$  such that

$$(f^{-1})^* \chi A \chi f^* = \text{Op}_h((f^{-1})^* a_{f,\chi}) + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

Here  $(f^{-1})^* a_{f,\chi} \in S_{1/2}^{\text{comp}}(\mathbb{R}^n) \cap C_0^\infty(T^*V_f)$  denotes the pullback of  $a_{f,\chi}$  under the map  $T^*V_f \rightarrow T^*U_f$  induced by  $f^{-1}$ . It can be seen from Lemma 5.1 that there exists a symbol  $a_f \in C^\infty(T^*U_f)$  such that for each  $\chi$ ,  $a_{f,\chi} = \chi^2 a_f + \mathcal{O}(h^{1/2} \tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}(X)}$  and moreover, the symbols  $a_f$  given by different coordinate charts agree modulo  $\mathcal{O}(h^{1/2} \tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}$ . This makes it possible to define the principal symbol map

$$\tilde{\sigma} : \Psi_{1/2}^{\text{comp}}(X) \rightarrow S_{1/2}^{\text{comp}}(X) / (h^{1/2} \tilde{h}^{1/2} S_{1/2}^{\text{comp}}(X)). \quad (5.4)$$

We will sometimes consider operators of the form  $f(h, \tilde{h})A$ , where  $f$  is some function and  $A \in \Psi_{1/2}^{\text{comp}}(X)$ . We put  $\tilde{\sigma}(fA) = f\tilde{\sigma}(A)$ ; it is defined modulo  $\mathcal{O}(f(h, \tilde{h})h^{1/2} \tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}(X)}$ . For instance, the symbol of an element of  $h^{1/2} \tilde{h}^{1/2} \Psi_{1/2}^{\text{comp}}(X)$  is defined modulo  $\mathcal{O}(h\tilde{h})_{S_{1/2}^{\text{comp}}(X)}$ .

The symbol map has a non-canonical right inverse  $\text{Op}_h : S_{1/2}^{\text{comp}}(X) \rightarrow \Psi_{1/2}^{\text{comp}}(X)$ , defined as follows: consider a locally finite covering of  $X$  by the domains  $U_j$  of some coordinate charts  $f_j : U_j \rightarrow V_j \subset \mathbb{R}^n$ , a partition of unity  $\chi_j \in C_0^\infty(U_j)$  on  $X$ , and some functions  $\chi'_j \in C_0^\infty(V_j)$  equal to 1 near  $f_j(\text{supp } \chi_j)$ . Then for  $a \in S_{1/2}^{\text{comp}}(X) \cap C_0^\infty(T^*X)$ , we put

$$\text{Op}_h(a) = \sum_j f_j^* \chi'_j \text{Op}_h((f_j^{-1})^*(\chi_j a)) \chi'_j (f_j^{-1})^*; \quad (5.5)$$

here  $(f_j^{-1})^*(\chi_j a) \in S_{1/2}^{\text{comp}}(\mathbb{R}^n) \cap C_0^\infty(T^*V_j)$  is quantized by (5.2). We have  $\text{Op}_h(a) \in \Psi_{1/2}^{\text{comp}}(X)$  and  $\tilde{\sigma}(\text{Op}_h(a)) = a + \mathcal{O}(h^{1/2} \tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}(X)}$ .

Using the properties of the  $S_{1/2}$  calculus on  $\mathbb{R}^n$  listed above, we get

**Lemma 5.2** (Properties of the  $\Psi_{1/2}$  calculus on manifolds). *Let  $X$  be a manifold. Then:*

1. *Each operator in  $\Psi_{1/2}^{\text{comp}}(X)$  is compactly microlocalized, pseudolocal, and has norm  $\mathcal{O}(1)$  in the sense of §3.1. If  $\text{supp } a \subset K$  for some compact set  $K \subset T^*X$  independent of  $h, \tilde{h}$ , then  $\text{WF}_h(\text{Op}_h(a)) \subset K$ .*
2. *The class  $\Psi_{1/2}^{\text{comp}}(X)$  of compactly microlocalized classical operators from §3.1 is contained in  $\Psi_{1/2}^{\text{comp}}(X)$ , with a correspondence between the symbol maps.*
3. *For  $A \in \Psi_{1/2}^{\text{comp}}(X)$ ,  $\tilde{\sigma}(A) = \mathcal{O}(h^{1/2} \tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}$  if and only if  $A \in (h^{1/2} \tilde{h}^{1/2}) \Psi_{1/2}^{\text{comp}}(X)$ .*

4. If  $A \in \Psi_{1/2}^{\text{comp}}(X)$ , then its adjoint  $A^*$  (with respect to some given density) also lies in  $\Psi_{1/2}^{\text{comp}}(X)$  and

$$\tilde{\sigma}(A^*) = \overline{\tilde{\sigma}(A)} + \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}.$$

5. If  $A, B \in \Psi_{1/2}^{\text{comp}}(X)$ , then  $AB \in \Psi_{1/2}^{\text{comp}}(X)$  and

$$\tilde{\sigma}(AB) = \tilde{\sigma}(A)\tilde{\sigma}(B) + \mathcal{O}(\tilde{h})_{S_{1/2}^{\text{comp}}}.$$

6. If  $A \in \Psi^k(X)$  and  $B \in \Psi_{1/2}^{\text{comp}}(X)$ , then  $AB, BA \in \Psi_{1/2}^{\text{comp}}(X)$ , and

$$\tilde{\sigma}(AB) = \sigma(A)\tilde{\sigma}(B) + \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}} = \tilde{\sigma}(BA). \quad (5.6)$$

Moreover  $[A, B] \in h^{1/2}\tilde{h}^{1/2}\Psi_{1/2}^{\text{comp}}(X)$ , and we have

$$\tilde{\sigma}([A, B]) = -ih\{\sigma(A), \tilde{\sigma}(B)\} + \mathcal{O}(h\tilde{h})_{S_{1/2}^{\text{comp}}}. \quad (5.7)$$

7. If  $A \in \Psi^k(X)$ ,  $B \in \Psi_{1/2}^{\text{comp}}(X)$ , and  $\{\sigma(A), \tilde{\sigma}(B)\} = \mathcal{O}(1)_{S_{1/2}^{\text{comp}}}$  (instead of  $\mathcal{O}((\tilde{h}/h)^{1/2})$  known a priori), then  $[A, B] \in h\Psi_{1/2}^{\text{comp}}(X)$ .

Note that in Lemma 5.2(1), we use the notion of wavefront set of a pseudolocal operator from §3.1, which gives a set independent of  $h$ . In particular, if  $a \in S_{1/2}^{\text{comp}}(X)$  is supported in an  $\mathcal{O}((h/\tilde{h})^{1/2})$  neighborhood of some compact set  $K$ , then  $\text{WF}_h(\text{Op}_h(a)) \subset K$ .

We also have the following version of the non-sharp Gårding inequality:

**Lemma 5.3.** Assume that  $X$  is a compact manifold,  $A \in \Psi_{1/2}^{\text{comp}}(X)$ ,  $\Psi_1 \in \Psi^{\text{comp}}(X)$ , and  $\text{Re } \tilde{\sigma}(A) > 0$  near  $\text{WF}_h(\Psi_1)$ . Then there exists a constant  $C$  such that for each  $u \in L^2(X)$ ,

$$\text{Re} \langle A\Psi_1 u, \Psi_1 u \rangle \geq C^{-1} \|\Psi_1 u\|_{L^2(X)}^2 - \mathcal{O}(h^\infty) \|u\|_{L^2(X)}^2.$$

*Proof.* For  $C > 0$  large enough, we can write

$$\text{Re } \tilde{\sigma}(A) = C^{-1} + |b|^2 + \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}} \text{ near } \text{WF}_h(\Psi_1)$$

for some  $b \in S_{1/2}^{\text{comp}}(X)$ . Take  $B = \text{Op}_h(b) \in \Psi_{1/2}^{\text{comp}}$ ; then

$$\text{Re } A = C^{-1} + B^* B + \mathcal{O}(\tilde{h})_{\Psi_{1/2}^{\text{comp}}} \text{ microlocally near } \text{WF}_h(\Psi_1).$$

Therefore,

$$\text{Re} \langle A\Psi_1 u, \Psi_1 u \rangle = C^{-1} \|\Psi_1 u\|_{L^2}^2 + \|B\Psi_1 u\|_{L^2}^2 + \mathcal{O}(\tilde{h}) \|\Psi_1 u\|_{L^2}^2 + \mathcal{O}(h^\infty) \|u\|_{L^2}^2.$$

The second term on the right-hand side is nonnegative; it remains to take  $\tilde{h}$  small enough so that the third term is absorbed by the first one.  $\square$

We will additionally need to work with symbols all of whose derivatives grow according to (5.1), but the symbols themselves grow like  $\log(1/h)$ ; those are the growth conditions satisfied by the logarithmically flattened escape function from Lemma 7.1. We need to know that certain properties of the  $\Psi_{1/2}$  calculus still hold in this setting:

**Lemma 5.4.** *Assume that  $a \in C_0^\infty(T^*X)$  satisfies*

$$a = \mathcal{O}(\log(1/h)); \quad \partial_{x,\xi}^\alpha a = \mathcal{O}((h/\tilde{h})^{-|\alpha|/2}), \quad |\alpha| > 0. \quad (5.8)$$

*In particular,  $a \in \log(1/h)S_{1/2}^{\text{comp}}(X)$ . Let  $A = \text{Op}_h(a) \in \log(1/h)\Psi_{1/2}^{\text{comp}}(X)$  be defined by (5.5). Then:*

1. *The symbol  $\tilde{\sigma}(A)$  from (5.4) is defined modulo  $\mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}$  (without the  $\log(1/h)$  factor), and  $\tilde{\sigma}(A) = a + \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}$ .*
2. *If  $B = \text{Op}_h(b)$  for some  $b$  satisfying (5.8), then  $[A, B] = \mathcal{O}(\tilde{h})_{\Psi_{1/2}^{\text{comp}}}$ .*
3. *If  $B \in \Psi^k(X)$ , then  $[A, B] \in h^{1/2}\tilde{h}^{1/2}\Psi_{1/2}^{\text{comp}}(X)$  and (5.7) holds.*

*Proof.* The remainder estimates of the asymptotic decompositions for the  $S_{1/2}(\mathbb{R}^n)$  calculus in (5.3) and [WuZw11, Lemma 3.3] depend on bounds on derivatives of the symbols involved, but not on the size of the symbols themselves. Therefore, the remainders in parts 4–6 of Lemma 5.1 are the same for symbols satisfying (5.8) as for  $S_{1/2}^{\text{comp}}$ . Note also that the class of symbols satisfying (5.8) is invariant under changes of variables and multiplication by symbols in  $S_{\text{cl}}^k(X)$  (though not by  $S_{1/2}^{\text{comp}}(X)$ ). All the statements above can now be verified in a straightforward fashion.  $\square$

Finally, we establish the following analog of Egorov's Theorem, needed in §6.1. A direct argument involving (3.6) and the method of stationary phase would give an  $\mathcal{O}(\tilde{h})$  error; the slightly more delicate argument below yields an  $\mathcal{O}(h^{1/2}\tilde{h}^{1/2})$  error for the Weyl quantization, giving a natural generalization (and a different proof) of [WuZw11, Lemma 3.3].

**Lemma 5.5.** *Let  $X_1, X_2$  be two manifolds of the same dimension,  $\varkappa$  a symplectomorphism mapping an open subset of  $T^*X_1$  onto an open subset of  $T^*X_2$ , and  $B : C^\infty(X_2) \rightarrow C^\infty(X_1)$  a compactly microlocalized semiclassical Fourier integral operator associated to  $\varkappa$ , in the sense of §3.2. Assume that  $A_j \in \Psi_{1/2}^{\text{comp}}(X_j)$  are such that*

$$\tilde{\sigma}(A_1) = \tilde{\sigma}(A_2) \circ \varkappa + \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}$$

*near the projection of  $\text{WF}_h(B)$  onto  $T^*X_1$ . Then*

$$A_1 B = B A_2 + \mathcal{O}(h^{1/2}\tilde{h}^{1/2});$$

*here  $\mathcal{O}(h^{1/2}\tilde{h}^{1/2})$  is understood in the sense of (3.2), as both sides of the equation are compactly microlocalized.*

*Proof.* Using a microlocal partition of unity, we may assume that  $A_1$  is microlocalized in a small neighborhood of some  $(x_1, \xi_1) \in T^*X_1$  and  $A_2$  is microlocalized in a small neighborhood of  $\varkappa(x_1, \xi_1)$ . We can then let  $X_1 = X_2 = \mathbb{R}^n$  and quantize  $\varkappa$  near  $\text{WF}_h(A_1) \times \text{WF}_h(A_2)$  by the unitary operators  $(B_1, B_1^{-1})$  constructed at the end of §3.2, multiplied by certain pseudodifferential cutoffs. We will use the families of symplectomorphisms  $\varkappa_t$ , Fourier integral operators  $B_t$ , and pseudodifferential operators  $Z_t$  from this construction.



Using the composition property (4) of Fourier integral operators in §3.2, we see that  $C_1 = BB_1^{-1}$  and  $C_2 = B_1^{-1}B$  lie in  $\Psi^{\text{comp}}$  and by the standard Egorov property (6) in §3.2,  $\sigma(C_1) = \sigma(C_2) \circ \varkappa$ . We then need to prove that

$$A_1C_1 = B_1(C_2A_2)B_1^{-1} + \mathcal{O}(h^{1/2}\tilde{h}^{1/2}), \quad (5.9)$$

where we know that

$$\tilde{\sigma}(A_1C_1) = \tilde{\sigma}(C_2A_2) \circ \varkappa + \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}. \quad (5.10)$$

Let  $A(t) = \text{Op}_h(a(t))$ , where

$$a(t) = \tilde{\sigma}(A_1C_1) \circ \varkappa_t^{-1} \quad (5.11)$$

and  $\text{Op}_h$  is defined by (5.2). Then  $A(0) = A_1C_1 + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ ; by (5.10),  $A(1) = C_2A_2 + \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{\Psi_{1/2}^{\text{comp}}}$ , and (5.9) reduces to

$$B_1A(1)B_1^{-1} = A(0) + \mathcal{O}(h^{1/2}\tilde{h}^{1/2}). \quad (5.12)$$

Using (3.8), we get

$$hD_t(B_tA(t)B_t^{-1}) = B_t([Z_t, A(t)] + hD_tA(t))B_t^{-1}. \quad (5.13)$$

Now, by part 5 of Lemma 5.1 (which is where we need the Weyl quantization),

$$[Z_t, A(t)] + hD_tA(t) = \frac{h}{i} \text{Op}_h(\{z_t, a(t)\} + \partial_t a(t)) + \mathcal{O}(h^{3/2}\tilde{h}^{3/2});$$

by (5.11), we get  $\{z_t, a(t)\} + \partial_t a(t) = \partial_t(a(t) \circ \varkappa_t) \circ \varkappa_t^{-1} = 0$  and thus the right-hand side of (5.13) is  $\mathcal{O}(h^{3/2}\tilde{h}^{3/2})$ . Integrating in  $t$  from 0 to 1, we get (5.12).  $\square$

**5.2. Second microlocalization.** In this subsection, we study certain operators microlocalized  $\mathcal{O}(h/\tilde{h})$  close to the energy surface  $p^{-1}(0)$ . (See §5.1 for why one needs the second semiclassical parameter  $\tilde{h}$ ; as always in this paper, we assume that  $\tilde{h}$  is small and  $h$  is small enough depending on  $\tilde{h}$ .) We need the operator  $A$  from Lemma 2.2 to be localized  $\mathcal{O}(h/\tilde{h})$  close to the energy surface to be able to approximate it by an operator of rank  $\mathcal{O}(h^{-\nu})$ . Without this additional localization, the rank of  $A_R$  from Lemma 2.2, and thus the number of resonances, would be estimated by  $\mathcal{O}(h^{-\nu-1})$ ; however, this estimate would be valid in an  $o(1)$  spectral window (with the imaginary part of the resonances still bounded by  $C_0h$ ) instead of the  $\mathcal{O}(h)$  one that we study.

Since  $h/\tilde{h} \ll h^{1/2}$ , operators microlocalized  $\mathcal{O}(h/\tilde{h})$  close to the energy surface will not be pseudodifferential even in the exotic classes studied in §5.1. In fact, they will not even be pseudolocal; their wavefront set will include transport along the Hamiltonian flow of  $p$  on the energy surface. This presents a difficulty with constructing a calculus of such operators; however, as shown in [SjZw, §5], one can still quantize symbols which are regular on the scale  $h/\tilde{h}$  in the direction transversal to the energy surface, on the scale 1 in the direction of the Hamiltonian flow of  $p$ , and on the scale  $(h/\tilde{h})^{1/2}$  in all other directions.

The second microlocal calculus of [SjZw] is rather involved and we do not use it here, proceeding instead as follows. Let  $\hat{p}$  be a real-valued symbol with

$$\hat{p} = p \quad \text{near the trapped set } K,$$

and which is elliptic near fiber infinity. We quantize  $\hat{p}$  to a self-adjoint operator  $\hat{P}$  (on the compact manifold  $X$  introduced in §4.1), and use an operator of the form  $\chi((\tilde{h}/h)\hat{P})$ , where  $\chi \in C_0^\infty$ ; this operator is microlocalized  $\mathcal{O}(\tilde{h}/h)$  close to the surface  $\hat{p}^{-1}(0)$ . In this subsection, we use spectral theory to get estimates on the resulting operator in an abstract setting; we will apply these to our problem in §6.1 and Lemma 7.8.

Throughout this subsection,  $X$  is a compact manifold without boundary and with a prescribed volume form, and  $\hat{P} \in \Psi^k(X)$ ,  $k > 0$ , is a symmetric pseudodifferential operator. Let  $\hat{p}$  be the principal symbol of  $\hat{P}$  (not to be confused with the notation for  $\langle \xi \rangle^{-2}p$  used in [Va11]). Assume that  $\hat{p}$  is elliptic near the fiber infinity; namely, the characteristic set  $\{\langle \xi \rangle^{-k}\hat{p} = 0\}$  does not intersect  $S^*X$  (and thus can be written  $\hat{p}^{-1}(0)$ ). Then  $\hat{P}$  is self-adjoint on  $L^2(X)$  with domain  $H_h^k(X)$  and compact resolvent (see for example [Ta, §7.10]).

For any bounded Borel measurable function  $\chi$  on  $\mathbb{R}$ , define  $\chi((\tilde{h}/h)\hat{P})$  by spectral theory (see for example [Ta, Chapter 8]). This operator is bounded on  $H_h^s(X)$  for each  $s$ , uniformly in  $h, \tilde{h}$ . Indeed, this is true for  $s = 0$  by spectral theory, for  $s \in k\mathbb{Z}$  by commuting with  $i + \hat{P}$ , which is an isomorphism  $H_h^{s+k} \rightarrow H_h^s$  for all  $s$ , and for general  $s$  by interpolation. Note also that the  $H_h^s(X)$  operator norm of  $\chi((\tilde{h}/h)\hat{P})$  depends only on  $s, \hat{P}$ , and  $\sup |\chi|$ . In particular, the unitary operator  $e^{it(\tilde{h}/h)\hat{P}}$  is bounded on each  $H_h^s$  uniformly in  $t$ .

We first show that for  $\chi$  Schwartz, the operator  $\chi((\tilde{h}/h)\hat{P})$  is microlocalized on  $\hat{p}^{-1}(0)$ :

**Lemma 5.6.** *Let  $\chi \in \mathcal{S}(\mathbb{R})$ . Then  $\chi((\tilde{h}/h)\hat{P})$  is a compactly microlocalized operator of norm  $\mathcal{O}(1)$ , in the sense of §3.1, and*

$$\text{WF}_h(\chi((\tilde{h}/h)\hat{P})) \subset \hat{p}^{-1}(0) \times \hat{p}^{-1}(0).$$

*Proof.* It suffices to show that if  $\Psi_1 \in \Psi^l(X)$  satisfies  $\text{WF}_h(\Psi_1) \cap \hat{p}^{-1}(0) = \emptyset$ , then

$$\chi((\tilde{h}/h)\hat{P})\Psi_1 = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}, \quad \Psi_1\chi((\tilde{h}/h)\hat{P}) = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

We prove the first statement. Take a large positive integer  $N$ . Since  $\hat{P}$  is elliptic near  $\text{WF}_h(\Psi_1)$ , there exists  $\Psi_2 \in \Psi^{l-kN}(X)$  such that  $\Psi_1 = \hat{P}^N \Psi_2 + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ . We then have

$$\chi((\tilde{h}/h)\hat{P})\Psi_1 = h^N \tilde{h}^{-N} \chi_N((\tilde{h}/h)\hat{P})\Psi_2 + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

Here  $\chi_N(\lambda) = \lambda^N \chi(\lambda)$  is Schwartz. The first term on the right-hand side is  $\mathcal{O}(h^{N-1})_{H_h^s \rightarrow H_h^{s+kN-l}}$  for all  $s$ , for  $h$  small enough depending on  $\tilde{h}$  (for example, for  $h < e^{-1/\tilde{h}}$ ); it remains to let  $N$  go to infinity.  $\square$

To establish further properties of  $\chi((\tilde{h}/h)\hat{P})$ , we use the following

**Lemma 5.7.** *Let  $\chi \in \mathcal{S}(\mathbb{R})$ , and  $B : C^\infty(X) \rightarrow C^\infty(X)$  be a polynomially bounded operator in the sense of §3.1. Then for each  $s, s'$  and each integer  $N \geq 0$ ,*

$$\begin{aligned} \chi((\tilde{h}/h)\hat{P})B &= \sum_{0 \leq j < N} \frac{(\tilde{h}/h)^j}{j!} (\text{ad}_{\hat{P}}^j B) \chi^{(j)}((\tilde{h}/h)\hat{P}) \\ &\quad + \mathcal{O}((\tilde{h}/h)^N \|\text{ad}_{\hat{P}}^N B\|_{H_h^s \rightarrow H_h^{s'}})_{H_h^s \rightarrow H_h^{s'}}. \end{aligned} \quad (5.14)$$

Here  $\chi^{(j)}$  denotes  $j$ -th derivative of  $\chi$  and  $\text{ad}_{\hat{P}} A = [\hat{P}, A]$  for any  $A$ . The constant in  $\mathcal{O}(\cdot)$  depends on  $\chi, N, s, s', \hat{P}$ , but not on  $B, h, \tilde{h}$ .

*Proof.* By the Fourier inversion formula,

$$\chi((\tilde{h}/h)\hat{P}) = \frac{1}{2\pi} \int \hat{\chi}(t) e^{it(\tilde{h}/h)\hat{P}} dt. \quad (5.15)$$

Here  $\hat{\chi} \in \mathcal{S}(\mathbb{R})$  is the Fourier transform of  $\chi$ . Now, for each  $j$ ,

$$\partial_t^j (e^{it(\tilde{h}/h)\hat{P}} B e^{-it(\tilde{h}/h)\hat{P}}) = (i\tilde{h}/h)^j e^{it(\tilde{h}/h)\hat{P}} (\text{ad}_{\hat{P}}^j B) e^{-it(\tilde{h}/h)\hat{P}};$$

since  $e^{\pm it(\tilde{h}/h)\hat{P}}$  is bounded uniformly in  $t$  on each  $H_h^s(X)$ , we have by Taylor's formula

$$\left\| e^{it(\tilde{h}/h)\hat{P}} B e^{-it(\tilde{h}/h)\hat{P}} - \sum_{0 \leq j < N} \frac{(it\tilde{h}/h)^j}{j!} \text{ad}_{\hat{P}}^j B \right\|_{H_h^s \rightarrow H_h^{s'}} \leq C |t\tilde{h}/h|^N \|\text{ad}_{\hat{P}}^N B\|_{H_h^s \rightarrow H_h^{s'}}.$$

It remains to multiply the operator in the left-hand side by  $e^{it(\tilde{h}/h)\hat{P}}$  on the right and substitute into (5.15).  $\square$

The operator  $\chi((\tilde{h}/h)\hat{P})$  is not  $h$ -pseudodifferential. As mentioned in the beginning of the subsection, we expect it to have nonlocal contributions corresponding to transport along the Hamiltonian flow for all times. To see this, recall (5.15) and the fact that  $e^{it(\tilde{h}/h)\hat{P}}$  is a Fourier integral operator associated to the Hamiltonian flow of  $\hat{p}$  at time  $t\tilde{h}$  (see also [SjZw, §5.4]). However, the nonlocal part of  $\chi((\tilde{h}/h)\hat{P})$  decays rapidly with respect to  $\tilde{h}$ , and commuting with certain pseudodifferential operators produces a power of  $\tilde{h}$ :

**Lemma 5.8.** *Let  $\chi \in \mathcal{S}(\mathbb{R})$ . Then:*

- (1) *if  $\Psi_1, \Psi_2 \in \Psi^0(X)$  have  $\text{WF}_h(\Psi_1) \cap \text{WF}_h(\Psi_2) = \emptyset$ , then*

$$\Psi_1 \chi((\tilde{h}/h)\hat{P}) \Psi_2 = \mathcal{O}(\tilde{h}^\infty);$$

- (2) *if  $\Psi_1 \in \Psi^0(X)$ , or both  $\Psi_1 \in \Psi_{1/2}^{\text{comp}}(X)$  and  $H_{\hat{p}} \tilde{\sigma}(\Psi_1) = \mathcal{O}(1)_{S_{1/2}^{\text{comp}}}$ , then*

$$[\chi((\tilde{h}/h)\hat{P}), \Psi_1] = \mathcal{O}(\tilde{h}).$$

In both cases, the  $\mathcal{O}(\cdot)$  is understood in the sense of (3.2), as the left-hand sides are compactly microlocalized by Lemma 5.6.

*Proof.* (1) We apply Lemma 5.7 with  $B = \Psi_2$ . Since  $\text{ad}_{\widehat{P}}^N \Psi_2 = \mathcal{O}(h^N)_{\Psi^{N(k-1)}}$ , we have for each  $N$  and each  $s$ ,

$$\Psi_1 \chi((\tilde{h}/h)\widehat{P}) \Psi_2 = \sum_{0 \leq j < N} \frac{(\tilde{h}/h)^j}{j!} \Psi_1(\text{ad}_{\widehat{P}}^j \Psi_2) \chi^{(j)}((\tilde{h}/h)\widehat{P}) + \mathcal{O}(\tilde{h}^N)_{H_h^s \rightarrow H_h^{s+N(1-k)}};$$

each term in the sum is  $\mathcal{O}(h^\infty)_{\Psi^{-\infty}}$  as  $\Psi_1(\text{ad}_{\widehat{P}}^j \Psi_2) = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ . It remains to let  $N \rightarrow \infty$ .

(2) We have  $[\widehat{P}, \Psi_1] = \mathcal{O}(h)_{L^2 \rightarrow L^2}$  (see part 7 of Lemma 5.2 for the second case); it remains to apply Lemma 5.7 with  $B = \Psi_1$  and  $N = 1$ .  $\square$

Finally, we establish a version of Egorov's theorem, needed in §6.1.

**Lemma 5.9.** *Let  $X_1, X_2$  be two compact manifolds of the same dimension,  $\varkappa$  be a symplectomorphism mapping an open subset of  $T^*X_1$  onto an open subset of  $T^*X_2$ , and  $B : C^\infty(X_2) \rightarrow C^\infty(X_1)$  be a compactly microlocalized semiclassical Fourier integral operator associated to  $\varkappa$ , in the sense of §3.2. Assume that  $\widehat{P}_j \in \Psi^{k_j}(X_j)$ ,  $k_j > 0$ , are symmetric operators elliptic near  $S^*X_j$  and*

$$\sigma(\widehat{P}_1) = \sigma(\widehat{P}_2) \circ \varkappa$$

*near the projection of  $\text{WF}_h(B)$  onto  $T^*X_1$ . Then for each  $\chi \in \mathcal{S}(\mathbb{R})$ ,*

$$\chi((\tilde{h}/h)\widehat{P}_1)B = B\chi((\tilde{h}/h)\widehat{P}_2) + \mathcal{O}(\tilde{h}),$$

*with  $\mathcal{O}(\tilde{h})$  understood in the sense of (3.2).*

*Proof.* As in the proof of Lemma 5.7, we use the Fourier inversion formula:

$$\chi((\tilde{h}/h)\widehat{P}_1)B - B\chi((\tilde{h}/h)\widehat{P}_2) = \frac{1}{2\pi} \int \hat{\chi}(t)(e^{it(\tilde{h}/h)\widehat{P}_1}B - Be^{it(\tilde{h}/h)\widehat{P}_2}) dt.$$

It is then enough to prove that

$$e^{it(\tilde{h}/h)\widehat{P}_1}B - Be^{it(\tilde{h}/h)\widehat{P}_2} = \mathcal{O}(t\tilde{h})_{L^2 \rightarrow L^2}.$$

Multiply the left-hand side by  $e^{-it(\tilde{h}/h)\widehat{P}_2}$  on the right and differentiate in  $t$ : we get

$$i(\tilde{h}/h)e^{it(\tilde{h}/h)\widehat{P}_1}(\widehat{P}_1B - B\widehat{P}_2)e^{-it(\tilde{h}/h)\widehat{P}_2};$$

this expression is  $\mathcal{O}(\tilde{h})_{L^2 \rightarrow L^2}$  uniformly in  $t$  by the standard Egorov property (6) in §3.2, as  $\widehat{P}_1B - B\widehat{P}_2 = \mathcal{O}(h)$ . It remains to integrate in  $t$ .  $\square$

## 6. APPROXIMATION BY FINITE RANK OPERATORS

In this section, we prove the following analog of [SjZw, Proposition 5.10]:

**Lemma 6.1.** *Let  $X$  be a compact manifold,  $\hat{P} \in \Psi^k(X)$ ,  $k > 0$ , a symmetric operator with principal symbol  $\hat{p} = \sigma(\hat{P})$  elliptic outside of a compact set, and  $\tilde{A} \in \Psi_{1/2}^{\text{comp}}(X)$  is such that  $\hat{p}$  has no critical points on  $\hat{p}^{-1}(0) \cap \text{WF}_h(\tilde{A})$ . Assume moreover that  $\tilde{\sigma}(\tilde{A}) = \tilde{a} + \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}}$ , where  $\tilde{a} \in C_0^\infty(T^*X) \cap S_{1/2}^{\text{comp}}(X)$  and there exists a constant  $\nu \geq 0$  such that for each  $R > 0$ , there exists  $C > 0$  such that*

$$\begin{aligned} & \text{Vol}_{\hat{p}^{-1}(0)}\{\exp(tH_{\hat{p}})(x, \xi) \mid |t| \leq R, \\ & (x, \xi) \in (\text{supp } \tilde{a} \cap \hat{p}^{-1}(0)) + B_{\hat{p}^{-1}(0)}(R(h/\tilde{h})^{1/2})\} \leq C(h/\tilde{h})^{n-1-\nu}. \end{aligned} \quad (6.1)$$

Here  $\text{Vol}_{\hat{p}^{-1}(0)}$  denotes the volume with respect to the Liouville measure on  $\hat{p}^{-1}(0)$  and  $V + B_{\hat{p}^{-1}(0)}(r)$  denotes the set of points in  $\hat{p}^{-1}(0)$  lying distance at most  $r$  away from  $V \subset \hat{p}^{-1}(0)$  (with respect to some fixed smooth metric). Finally, let  $\chi \in C_0^\infty(\mathbb{R})$  and define

$$A = \chi((\tilde{h}/h)\hat{P})\tilde{A}.$$

Then we can write  $A = A_R + A_E$ , where  $A_R, A_E : C^\infty(X) \rightarrow C^\infty(X)$  are compactly microlocalized and for some constant  $C(\tilde{h})$  independent of  $h$ ,

$$A_R = \mathcal{O}(1), \quad A_E = \mathcal{O}(\tilde{h}), \quad \text{rank } A_R \leq C(\tilde{h})h^{-\nu}.$$

Here  $\mathcal{O}(\cdot)$  is understood in the sense of (3.2).

Lemma 6.1 is the main component needed to approximate the operator  $A$  from Lemma 2.2 by a finite rank operator. In our case, the volume estimate (6.1) is a direct consequence of the definition of the upper Minkowski dimension of the trapped set (1.3) and the fact that the symbol  $\tilde{a}$  will be supported  $\mathcal{O}((h/\tilde{h})^{1/2})$  close to the trapped set. See §7.4 for details.

To prove Lemma 6.1, we will first, in §6.1, conjugate the operator  $A$  locally by a semiclassical Fourier integral operator to make  $\hat{p} = \xi_1$  thus trivializing the ‘second microlocalized’ factor  $\chi((\tilde{h}/h)\hat{P})$  of  $A$ . To simplify the discussion, we drop the second microlocalized factor in this paragraph and explain how to approximate  $\tilde{A}$  rather than  $A$ . For that, cover  $\text{supp } \tilde{a}$  by balls of size  $\sim (h/\tilde{h})^{1/2}$  (the analog of this step is carried out in §6.2); then to each such ball, associate an operator of rank  $\mathcal{O}(1)$  which is a function of a quantum harmonic oscillator, shifted to be centered in that ball (see §6.3). The sum  $\tilde{\Psi}_\Pi$  of these operators will be elliptic near  $\text{WF}_h(\tilde{A})$ , thus we can approximate  $\tilde{A}$  by a multiple of  $\tilde{\Psi}_\Pi$  (see §§6.3–6.4). However, the rank of  $\tilde{\Psi}_\Pi$  is bounded by a constant times the number of balls of size  $(h/\tilde{h})^{1/2}$  that are needed to cover  $\text{supp } \tilde{a}$ ; this number can be estimated by the volume in (6.1).

We generally follow the proof of [SjZw, Proposition 5.10] (see also [Zw, §6.4] for an application of some of the ideas used in a simpler setting), with the following two differences. First of all, we treat the operators microlocalized  $\mathcal{O}(h/\tilde{h})$  near the energy surface as in §5.2, rather than using [SjZw, §5]. Secondly, we prove in detail (see Lemma 6.2) that the operator  $\tilde{\Psi}_\Pi$  lies in  $\Psi_{1/2}^{\text{comp}}$ . This is not trivial because, even though the operator associated to each ball lies in  $\Psi_{1/2}^{\text{comp}}$ , we sum  $\sim (\tilde{h}/h)^{n-1-\nu}$  many such operators; this step is skipped in [SjZw].

One of the anonymous referees of this paper has suggested an alternative approach to this lemma. If one arranges that  $A$  is a positive operator, and shows that the trace of  $A$  is bounded by  $Ch^{-n}(h/\tilde{h})^{n-\nu}$ , or more generally by  $C(\tilde{h})h^{-\nu}$ , then  $A$  can have no more than  $C(\tilde{h})\tilde{h}^{-1}h^{-\nu}$  many eigenvalues greater than  $\tilde{h}$ , and the decomposition follows. If  $A$  were pseudodifferential, its trace could be computed by integrating its full symbol as in [DiSj, Theorem 9.4]. The factor  $\chi((\tilde{h}/h)\widehat{P})$  prevents one from applying this approach directly, but this difficulty could likely be overcome using the methods of §5.2.

We prove Lemma 6.1 over the course of the following four subsections:

**6.1. Reduction to a model problem.** We reduce the general case to the following model case:

$$X = X_M = \mathbb{S}_{x_1}^1 \times \mathbb{R}_{x'}^{n-1}, \quad \widehat{P} = hD_{x_1}.$$

The manifold  $X_M$  is not compact, so, strictly speaking, the statement of Lemma 6.1 does not apply. However, the only place where we use the compactness of  $X$  in the proof is the application of Lemma 5.9 in this subsection; since  $hD_{x_1}$  is self-adjoint on  $L^2(X_M)$  and the associated unitary operator  $e^{it(\tilde{h}/h)hD_{x_1}}$  is a shift in the  $x_1$  variable and thus bounded on each  $H_h^s(X_M)$  uniformly in  $t$ , the proof of this lemma still goes through. The operator  $\chi(\tilde{h}D_{x_1})$  is a Fourier series multiplier in the  $x_1$  variable; therefore, it is properly supported and polynomially bounded in the sense of §3.1; moreover, the product of this operator with a compactly microlocalized operator will also be compactly microlocalized.

We now construct finitely many operators  $\Psi_j \in \Psi^{\text{comp}}(X)$  such that

- (1)  $\sum_j \Psi_j = 1$  microlocally near  $\hat{p}^{-1}(0) \cap \text{WF}_h(\tilde{A})$ ;
- (2) for each  $j$ , there exists a symplectomorphism  $\varkappa_j$  from a neighborhood of  $\text{WF}_h(\Psi_j)$  onto some open subset of  $T^*X_M$  such that  $\hat{p} = \xi_1 \circ \varkappa_j$  on the domain of  $\varkappa_j$ ;
- (3) for each  $j$ , there exist compactly microlocalized semiclassical Fourier integral operators

$$B_j : C^\infty(X_M) \rightarrow C^\infty(X), \quad B'_j : C^\infty(X) \rightarrow C^\infty(X_M),$$

associated to  $\varkappa_j$  and  $\varkappa_j^{-1}$ , respectively, such that  $B_j B'_j = 1$  microlocally near  $\text{WF}_h(\Psi_j)$ .

Indeed, by the Darboux theorem (see for example [Zw, Theorem 12.1]) each  $(x, \xi) \in \hat{p}^{-1}(0) \cap \text{WF}_h(\tilde{A})$  has a neighborhood  $U_{(x, \xi)} \subset T^*X$  with a symplectomorphism  $\varkappa : U_{(x, \xi)} \rightarrow T^*X_M$  such that  $\hat{p} = \xi_1 \circ \varkappa_{(x, \xi)}$  on  $U_{(x, \xi)}$ . Using the method described at the end of §3.2, we can find compactly microlocalized semiclassical Fourier integral operators  $B_{(x, \xi)} : C^\infty(X_M) \rightarrow C^\infty(X)$ ,  $B'_{(x, \xi)} : C^\infty(X) \rightarrow C^\infty(X_M)$  quantizing  $\varkappa$  near the closure of  $V_{(x, \xi)} \times \varkappa_{(x, \xi)}(V_{(x, \xi)})$ , where  $V_{(x, \xi)} \subset U_{(x, \xi)}$  is some neighborhood of  $(x, \xi)$ . It remains to choose  $\Psi_j$  as a microlocal partition of unity subordinate to an open cover of  $\hat{p}^{-1}(0)$  by finitely many of the sets  $V_{(x, \xi)}$ .

By Lemma 5.9 (see the remark about non-compactness of  $X_M$  in the beginning of this subsection), we have

$$B'_j \chi((\tilde{h}/h)\hat{P}) = \chi(\tilde{h}D_{x_1})B'_j + \mathcal{O}(\tilde{h});$$

here  $\mathcal{O}(\tilde{h})$  is understood in the sense of (3.2), as both sides of the equation are compactly microlocalized. Next, let  $\chi_j \in C_0^\infty(T^*X)$  be supported inside the domain of  $\varkappa_j$ , but  $\chi_j = 1$  near the projection of  $\text{WF}_h(B'_j)$  onto  $T^*X$ , and let  $\tilde{A}_j \in \Psi_{1/2}^{\text{comp}}(X_M)$  satisfy

$$\tilde{\sigma}(\tilde{A}_j) = \tilde{a}_j + \mathcal{O}(h^{1/2}\tilde{h}^{1/2}), \quad \tilde{a}_j = (\chi_j \tilde{a}) \circ \varkappa_j^{-1};$$

here  $\tilde{a}_j$  is extended by zero outside of the image of  $\varkappa_j$ . By Lemma 5.5,

$$B'_j \tilde{A} = \tilde{A}_j B'_j + \mathcal{O}(h^{1/2}\tilde{h}^{1/2});$$

moreover,  $\tilde{a}_j$  satisfies the volume bound (6.1), with  $\xi_1$  taking the place of  $\hat{p}$ . By Lemmas 5.6 and 5.8(1),

$$\begin{aligned} A &= \sum_j \Psi_j \chi((\tilde{h}/h)\hat{P}) \tilde{A} + \mathcal{O}(\tilde{h}^\infty) = \sum_j \Psi_j B_j B'_j \chi((\tilde{h}/h)\hat{P}) \tilde{A} + \mathcal{O}(\tilde{h}^\infty) \\ &= \sum_j \Psi_j B_j \chi(\tilde{h}D_{x_1}) \tilde{A}_j B'_j + \mathcal{O}(\tilde{h}). \end{aligned}$$

It now suffices to establish the decomposition for each of the operators  $\chi(\tilde{h}D_{x_1})\tilde{A}_j$  (bearing in mind that  $B_j, B'_j$  have norm  $\mathcal{O}(1)$  — see property 5 in §3.2). Therefore, we henceforth assume that  $X = X_M$  and  $\hat{P} = hD_{x_1}$ .

**6.2. Covering by cylinders.** We now cover  $\text{supp } \tilde{a} \cap \hat{p}^{-1}(0)$  by cylinders.

We write  $(x, \xi) \in T^*X_M$  as  $(x_1, x', \xi_1, \xi')$ , where  $x_1 \in \mathbb{S}^1$ ,  $\xi_1 \in \mathbb{R}$ , and  $x', \xi' \in \mathbb{R}^{n-1}$ . The energy surface is  $\hat{p}^{-1}(0) = \{\xi_1 = 0\}$ . The Hamiltonian flow of  $\hat{p}$  is  $2\pi$ -periodic; thus, for given  $(x, \xi)$  and  $R \geq \pi$ , the set  $\{\exp(tH_{\hat{p}}(x, \xi)) \mid |t| \leq R\}$  is equal to the circle in the direction of the  $x_1$  variable passing through  $(x, \xi)$ . If  $V \subset T^*X_M$ , define

$$\Pi(V) = \{(x', \xi') \mid \exists x_1 : (x_1, x', 0, \xi') \in V\} \subset \mathbb{R}^{2(n-1)}. \quad (6.2)$$

The definition (6.2) is useful for handling the second microlocalization in §6.4. In fact, covering  $\Pi(V)$  by  $(h/\tilde{h})^{1/2}$  sized balls is morally the same as covering  $V \cap \hat{p}^{-1}([-h/\tilde{h}, h/\tilde{h}])$  (which is where the operator  $A$  is microlocalized) by cylinders of size  $h/\tilde{h}$  in the direction transversal to  $\hat{p}^{-1}(0)$ , of size 1 in the direction of the Hamiltonian flow of  $\hat{p}$ , and of size  $(h/\tilde{h})^{1/2}$  in all other directions; to each such cylinder will correspond an operator of rank  $\mathcal{O}(1)$ , which is essentially the product of a  $C_0^\infty$  function of  $\tilde{h}D_{x_1}$  and spectral projector associated to a shifted harmonic oscillator. This subsection and §6.3 will handle the  $(x', \xi')$  directions, constructing the covering by balls and the corresponding finite rank operators.

The volume estimate (6.1) with  $R = \pi$  implies (the choice of  $1/2$  is convenient later)

$$\text{Vol}_{x', \xi'} \left( \Pi(\text{supp } \tilde{a}) + B\left(0, \frac{1}{2}(h/\tilde{h})^{1/2}\right) \right) \leq C(h/\tilde{h})^{n-1-\nu};$$



here  $B(\rho, r)$  denotes the closed Euclidean ball of radius  $r$  centered at  $\rho$ . Following [Sj90, Lemma 3.3], take a maximal set of points

$$(x'_l, \xi'_l) \in \Pi(\text{supp } \tilde{a}), \quad 1 \leq l \leq M(h, \tilde{h}),$$

such that the distance between any two distinct points in this set is  $> (h/\tilde{h})^{1/2}$ . Then

$$\bigsqcup_{l=1}^{M(h, \tilde{h})} B\left((x'_l, \xi'_l), \frac{1}{2}(h/\tilde{h})^{1/2}\right) \subset \Pi(\text{supp } \tilde{a}) + B\left(0, \frac{1}{2}(h/\tilde{h})^{1/2}\right);$$

therefore, comparing the volumes of the two sides, we get

$$M(h, \tilde{h}) \leq C(h/\tilde{h})^{-\nu}. \quad (6.3)$$

However, we also know by maximality that

$$\Pi(\text{supp } \tilde{a}) \subset \bigcup_{l=1}^{M(h, \tilde{h})} B((x'_l, \xi'_l), (h/\tilde{h})^{1/2}). \quad (6.4)$$

**6.3. The finite rank operator.** To each ball in the covering, we associate a finite rank operator constructed using a function of a shifted quantum harmonic oscillator. The sum of these operators will generate the finite rank term in the decomposition. As noted in the beginning of this section, since the number of balls in the covering grows polynomially in  $h$ , we need to take care when summing up the corresponding operators.

Consider the following shifted harmonic oscillators on  $\mathbb{R}^{n-1}$  (here  $x'_{l,i}, \xi'_{l,i}$  are the  $i$ th coordinates of  $x'_l, \xi'_l$  respectively)

$$\tilde{P}_l = \sum_{i=1}^{n-1} (hD_{x'_i} - \xi'_{l,i})^2 + (x'_i - x'_{l,i})^2.$$

Let  $\tilde{\chi} \in C_0^\infty(-2, 2)$  be nonnegative and equal to 1 on  $(-1, 1)$  and put

$$\tilde{\Psi}_l = \tilde{\chi}((\tilde{h}/h)\tilde{P}_l).$$

**Lemma 6.2.** *Take  $\chi_b \in C_0^\infty(\mathbb{R}^{n-1})$  equal to 1 near the projection of  $\Pi(\text{supp } \tilde{a})$  onto the base space  $\mathbb{R}_{x'}^{n-1}$ . Then the sum*

$$\tilde{\Psi}_\Pi = \sum_{l=1}^{M(h, \tilde{h})} \chi_b(x') \tilde{\Psi}_l \chi_b(x')$$

*lies in  $\Psi_{1/2}^{\text{comp}}(\mathbb{R}^{n-1})$  (the cutoff  $\chi_b$  is needed because the operators  $\tilde{\Psi}_l$  are not properly supported), and its symbol  $\tilde{s}_\Pi = \tilde{\sigma}(\tilde{\Psi}_\Pi)$  satisfies*

$$|\tilde{s}_\Pi| \geq 1 - \mathcal{O}(\tilde{h})$$

*on  $\Pi(\text{supp } \tilde{a})$ . Moreover, if we consider  $\tilde{\Psi}_\Pi$  as an operator on  $L^2(\mathbb{R}^{n-1})$ , then*

$$\text{rank } \tilde{\Psi}_\Pi \leq C\tilde{h}^{\nu+1-n}h^{-\nu}. \quad (6.5)$$

For (6.5), we use (6.3) and the fact that the rank of each  $\tilde{\Psi}_l$  is bounded by  $C\tilde{h}^{1-n}$ . The latter fact follows from Weyl's law for the eigenvalues for the harmonic oscillator (see for example [Zw, Theorem 6.3]) and the fact that  $\tilde{\chi}$  is compactly supported.

Since functions of the quantum harmonic oscillator are not properly supported operators, and to facilitate the rescaling argument in the proof of Lemma 6.3, we use a variation of the  $\Psi_{1/2}^{\text{comp}}(\mathbb{R}^{n-1})$  calculus. Define the operator classes  $\hat{S}$  and  $\hat{S}_{1/2}$  on  $T^*\mathbb{R}^{n-1}$  as follows:

$$\begin{aligned} a(x', \xi'; h, \tilde{h}) \in \hat{S} &\iff \forall \alpha \forall N \sup_{(x', \xi') \in T^*\mathbb{R}^{n-1}} |\partial_{x', \xi'}^\alpha a| \leq C_{\alpha N} \langle (x', \xi') \rangle^{-N}; \\ a(x', \xi'; h, \tilde{h}) \in \hat{S}_{1/2} &\iff \forall \alpha \forall N \sup_{(x', \xi') \in T^*\mathbb{R}^{n-1}} |\partial_{x', \xi'}^\alpha a| \leq C_{\alpha N} (h/\tilde{h})^{-|\alpha|/2} \langle (x', \xi') \rangle^{-N}. \end{aligned} \quad (6.6)$$

Clearly  $\hat{S} \subset \hat{S}_{1/2}$ . The difference between  $\hat{S}_{1/2}$  and the class  $S_{1/2}^{\text{comp}}$  from §5.1 is that we do not require compact essential support, imposing instead uniform bounds on the derivatives of the symbol as  $(x', \xi') \rightarrow \infty$ . However,  $S_{1/2}^{\text{comp}}(\mathbb{R}^{n-1}) \not\subset \hat{S}_{1/2}$ , as the former does not require uniform bounds as  $x' \rightarrow \infty$ . For  $a \in \hat{S}_{1/2}$ , we define its Weyl quantization  $\widehat{\text{Op}}_h(a)$  as

$$\widehat{\text{Op}}_h(a)u(x') = (2\pi h)^{1-n} \int e^{\frac{i}{h}(x'-y') \cdot \xi'} a\left(\frac{x'+y'}{2}, \xi'\right) u(y') d\xi' dy'. \quad (6.7)$$

The difference from (5.2) is the lack of the cutoff  $\tilde{\chi}(x-y)$ ; because of this, the operator  $\widehat{\text{Op}}_h(a)$  need not be properly supported. However, this operator acts on the space of Schwartz functions on  $\mathbb{R}^{n-1}$ , as well as on  $L^2(\mathbb{R}^{n-1})$ , therefore one can still compose two such operators. The resulting calculus has the properties listed in Lemma 5.1, parts 2–5, with the improvement that  $\widehat{\text{Op}}_h(a)\widehat{\text{Op}}_h(b) = \widehat{\text{Op}}_h(a\#b)$  without the  $\mathcal{O}(h^\infty)$  remainder; in fact,  $\hat{S}_{1/2}$  lies in the class  $\tilde{S}_{1/2}(T^*\mathbb{R}^{n-1})$  from [WuZw11, (3.5)].

The proof of Lemma 6.2 is based on the following precise estimates on the full symbol of a function of the harmonic oscillator:

**Lemma 6.3.** *Put*

$$P_0(h) = -h^2 \Delta_{x'}^2 + |x'|^2,$$

*an unbounded operator on  $L^2(\mathbb{R}^{n-1})$ , and let  $\chi \in C_0^\infty(\mathbb{R})$ . Then:*

$$\begin{aligned} 1. \quad &\chi(P_0(h)) = \widehat{\text{Op}}_h(a_\chi), \text{ where } a_\chi(x', \xi'; h) \in \hat{S} \text{ and} \\ &a_\chi(x', \xi'; h) = \chi(|x'|^2 + |\xi'|^2) + \mathcal{O}(h)_{\hat{S}}, \\ &a_\chi = \mathcal{O}(h^\infty)_{\hat{S}} \text{ outside of any neighborhood of } \{(x', \xi') \mid |x'|^2 + |\xi'|^2 \in \text{supp } \chi\}. \end{aligned} \quad (6.8)$$

$$2. \quad \chi((\tilde{h}/h)P_0(h)) = \widehat{\text{Op}}_h(a_{\chi((\tilde{h}/h)\cdot)}), \text{ where } a_{\chi((\tilde{h}/h)\cdot)}(x', \xi'; h) \in \hat{S}_{1/2} \text{ and}$$

$$a_{\chi((\tilde{h}/h)\cdot)} = \chi((\tilde{h}/h)(|x'|^2 + |\xi'|^2)) + \mathcal{O}(\tilde{h})_{\hat{S}_{1/2}}.$$

*If  $T > 0$  satisfies  $\text{supp } \chi \subset (-T, T)$  and  $r = \sqrt{|x'|^2 + |\xi'|^2}$ , then for each  $\alpha, N$ ,*

$$|\partial_{x', \xi'}^\alpha a_{\chi((\tilde{h}/h)\cdot)}(x', \xi')| \leq C_{\alpha N} (h/\tilde{h})^{-|\alpha|/2} (h/r^2)^N \text{ if } r \geq (Th/\tilde{h})^{1/2}.$$

The proof of Lemma 6.3 is given at the end of this subsection. Using part 2 of it together with conjugating by an exponential and performing a shift to reduce to the case  $(x'_l, \xi'_l) = 0$ , we see that  $\tilde{\Psi}_l = \widehat{\text{Op}}_h(\check{s}_l)$ , where

$$\check{s}_l(x', \xi'; h, \tilde{h}) = a_{\tilde{\chi}((\tilde{h}/h)\cdot)}(x' - x'_l, \xi' - \xi'_l; h)$$

lies in  $\widehat{S}_{1/2}$  uniformly in  $l$ . Moreover, putting  $T = 2$  in part 2 of Lemma 6.3 and recalling that  $\text{supp } \tilde{\chi} \subset (-2, 2)$ , we get

$$\check{s}_l(x', \xi') = \tilde{\chi}((\tilde{h}/h)(|x' - x'_l|^2 + |\xi' - \xi'_l|^2)) + \mathcal{O}(\tilde{h})_{\widehat{S}_{1/2}}, \quad (6.9)$$

$$\check{s}_l = \mathcal{O}((h/(|x' - x'_l|^2 + |\xi' - \xi'_l|^2))^\infty)_{\widehat{S}_{1/2}} \text{ outside of } B((x'_l, \xi'_l), 2(h/\tilde{h})^{1/2}). \quad (6.10)$$

*Proof of Lemma 6.2.* The idea is to use the improved bound on the full symbol (6.10) together with information on how many of the points  $(x'_l, \xi'_l)$  can lie close to some fixed point. Fix  $(x'_0, \xi'_0) \in T^*\mathbb{R}^{n-1}$  and consider the dyadic partition

$$\{1, \dots, M(h, \tilde{h})\} = \bigsqcup_{j \geq 0} L_j,$$

$$L_0 = \{l \mid (x'_l, \xi'_l) \in B((x'_0, \xi'_0), 2(h/\tilde{h})^{1/2})\},$$

$$L_j = \{l \mid (x'_l, \xi'_l) \in B((x'_0, \xi'_0), 2^{j+1}(h/\tilde{h})^{1/2}) \setminus B((x'_0, \xi'_0), 2^j(h/\tilde{h})^{1/2})\}, \quad j \geq 1.$$

By the triangle inequality,

$$\bigcup_{l \in L_j} B\left((x'_l, \xi'_l), \frac{1}{2}(h/\tilde{h})^{1/2}\right) \subset B((x'_0, \xi'_0), 2^{j+2}(h/\tilde{h})^{1/2}).$$

Moreover, since the pairwise distance between the points  $(x'_l, \xi'_l)$  is  $> (h/\tilde{h})^{1/2}$  by the construction in §6.2, the union on the left-hand side is disjoint. By comparing the volumes of the two sets, we arrive at the bound

$$|L_j| \leq C \cdot 2^{j(2n-2)}. \quad (6.11)$$

Now, we write  $\tilde{\Psi}_\Pi = \chi_b \widehat{\text{Op}}_h(\check{s}_\Pi) \chi_b$ , where

$$\check{s}_\Pi = \sum_{j \geq 0} \check{s}_\Pi^{(j)}, \quad \check{s}_\Pi^{(j)} = \sum_{l \in L_j} \check{s}_l.$$

The symbol  $\check{s}_\Pi^{(0)}$  will be the principal part of  $\check{s}_\Pi$  at  $(x'_0, \xi'_0)$ . The symbols  $\check{s}_\Pi^{(j)}$  for  $j \geq 1$  at  $(x'_0, \xi'_0)$  are nonzero because of tunneling; however, we can estimate their contribution as  $\mathcal{O}(\tilde{h}^\infty)$ . More precisely, we combine (6.10) with the bound (6.11) on  $|L_j|$  to get

$$\check{s}_\Pi^{(j)} = \mathcal{O}((2^{-j}\tilde{h})^\infty)_{\widehat{S}_{1/2}} \text{ at } (x'_0, \xi'_0) \text{ for } j \geq 1.$$

Hence  $\sum_{j=1}^\infty \check{s}_\Pi^{(j)}$  is  $\mathcal{O}(\tilde{h}^\infty)_{\widehat{S}_{1/2}}$  at  $(x'_0, \xi'_0)$ . Now, by (6.11),  $\check{s}_\Pi^{(0)}$  is the sum of a bounded number of  $\check{s}_l$ 's and is therefore in  $\widehat{S}_{1/2}$  at  $(x'_0, \xi'_0)$ . Moreover, if  $(x'_0, \xi'_0) \in \Pi(\text{supp } \tilde{a})$ , then by (6.4) and (6.9), at least one term in the sum for  $\check{s}_\Pi^{(0)}(x'_0, \xi'_0)$  is equal to  $1 + \mathcal{O}(\tilde{h})$ , and the

other terms are nonnegative modulo  $\mathcal{O}(\tilde{h})$ . Therefore,  $\tilde{s}_\Pi \in \widehat{S}_{1/2}$  and  $|\tilde{s}_\Pi| \geq 1 - \mathcal{O}(\tilde{h})$  near  $\Pi(\text{supp } \tilde{a})$ . Also,  $\tilde{s}_\Pi = \mathcal{O}(h^\infty)_{\widehat{S}_{1/2}}$  outside a fixed compact set. Then  $\tilde{\Psi}_\Pi = \chi_b \widehat{\text{Op}}_h(\tilde{s}_\Pi) \chi_b$  lies in  $\Psi_{1/2}^{\text{comp}}(\mathbb{R}^{n-1})$  and  $\tilde{\sigma}(\tilde{\Psi}_\Pi) = \chi_b(x')^2 \tilde{s}_\Pi + \mathcal{O}(h^{1/2} \tilde{h}^{1/2})_{S_{1/2}^{\text{comp}}(\mathbb{R}^{n-1})}$ ; this finishes the proof.  $\square$

*Proof of Lemma 6.3.* 1. This follows from standard results on functional calculus of pseudodifferential operators, see for example [DiSj, Chapter 8]. In particular, the fact that  $a \in \widehat{S}$  follows from [DiSj, Theorem 8.7], with the order function  $m = 1 + |x'|^2 + |\xi'|^2$ , while (6.8) follows from the expansion for  $a_\chi$  preceding [DiSj, (8.15)]. The validity of (6.8) in  $\widehat{S}$  is checked as in the proof of [DiSj, Theorem 8.7], by using the composition formula and the fact that  $\chi(P_0(h)) = \chi_k(P_0(h))(P_0(h) + i)^{-k}$ , where  $\chi_k(\lambda) = (\lambda + i)^k \chi(\lambda)$  lies in  $C_0^\infty(\mathbb{R})$  and  $(P_0(h) + i)^{-k}$  has symbol in  $S(m^{-k})$ , in the notation of [DiSj].

2. We use the unitary rescaling operator (see [Zw, §6.1.2])

$$T_\beta : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\mathbb{R}^{n-1}), \quad \beta > 0, \quad (T_\beta u)(\tilde{x}) = \beta^{(n-1)/4} u(\beta^{1/2} \tilde{x});$$

then

$$\chi((h/\tilde{h})P_0(h)) = T_\beta^{-1} \chi((\tilde{h}\beta/h)P_0(h/\beta)) T_\beta.$$

Moreover, the operator  $T_\beta$  changes Weyl quantized symbols as follows:

$$\widehat{\text{Op}}_h(a) = T_\beta^{-1} \widehat{\text{Op}}_{h/\beta}(a_\beta) T_\beta, \quad a_\beta(\tilde{x}, \tilde{\xi}) = a(\beta^{1/2} \tilde{x}, \beta^{1/2} \tilde{\xi}).$$

Take  $\beta = h/\tilde{h}$ ; then

$$\begin{aligned} \chi((\tilde{h}/h)P_0(h)) &= T_\beta^{-1} \chi(P_0(\tilde{h})) T_\beta = T_\beta^{-1} \widehat{\text{Op}}_{\tilde{h}}(a_\chi(\cdot, \cdot; \tilde{h})) T_\beta = \widehat{\text{Op}}_h(a_{\chi((\tilde{h}/h)\cdot)}), \\ a_{\chi((\tilde{h}/h)\cdot)}(x', \xi'; h) &= a_\chi((\tilde{h}/h)^{1/2} x', (\tilde{h}/h)^{1/2} \xi'; \tilde{h}). \end{aligned}$$

It remains to use the estimates on  $a_\chi$  from part 1, with  $\tilde{h}$  taking the place of  $h$ .  $\square$

**6.4. Approximation.** We finally use parametrices to obtain the approximation.

We write  $\tilde{A} = \tilde{A}' + \tilde{A}''$ , where

$$\tilde{A}' = \text{Op}_h(\tilde{a}'), \quad \tilde{a}'(x_1, x', \xi_1, \xi') = \tilde{\chi}(\xi_1) \tilde{a}(x_1, x', 0, \xi'),$$

with  $\tilde{\chi} \in C_0^\infty(\mathbb{R})$  as in §6.3. Consider the operator  $\tilde{\Psi}_\Pi$  from Lemma 6.2, take  $\tilde{\chi}' \in C_0^\infty(\mathbb{R})$  equal to 1 near  $\text{supp } \tilde{\chi}$ , and put

$$\Psi_\Pi = \tilde{\chi}'(hD_{x_1}) \otimes \tilde{\Psi}_\Pi \in \Psi_{1/2}^{\text{comp}}(X_M).$$

The symbol  $s_\Pi = \tilde{\sigma}(\Psi_\Pi)$  satisfies  $|s_\Pi| \geq 1 - \mathcal{O}(\tilde{h})$  on  $\text{supp}(\tilde{a}')$ . (Here we use that  $|\tilde{s}_\Pi| \geq 1 - \mathcal{O}(\tilde{h})$  on  $\Pi(\text{supp } \tilde{a})$  from Lemma 6.2, the definition (6.2) of  $\Pi$ , and  $\tilde{\chi}'\tilde{\chi} = \tilde{\chi}$ .) Put  $B' = \text{Op}_h(\tilde{a}'/s_\Pi) \in \Psi_{1/2}^{\text{comp}}(X_M)$ ; then

$$\tilde{A}' = \Psi_\Pi B' + \mathcal{O}(\tilde{h})_{\Psi_{1/2}^{\text{comp}}(X_M)},$$

and we have written  $\tilde{A}'$  as the sum of a finite rank term and an  $\mathcal{O}(\tilde{h})$  remainder, as needed in the statement of Lemma 6.1. To treat the  $\tilde{A}''$  term, put  $B'' = \text{Op}_h((\tilde{a} - \tilde{a}')/\xi_1) \in$

$(\tilde{h}/h)^{1/2}\Psi_{1/2}^{\text{comp}}(X_M)$  (with the prefactor coming from the fact that  $\partial_{\xi_1}\tilde{a}$  can grow like  $(\tilde{h}/h)^{1/2}$ ). Then by part 6 of Lemma 5.2,

$$\tilde{A}'' = (hD_{x_1})B'' + \mathcal{O}(\tilde{h})_{\Psi_{1/2}^{\text{comp}}(X_M)}.$$

Therefore (with  $\mathcal{O}(\tilde{h})$  below in the sense of (3.2))

$$A = \chi(\tilde{h}D_{x_1})\tilde{A} = \chi(\tilde{h}D_{x_1})\Psi_{\Pi}B' + \chi(\tilde{h}D_{x_1})(hD_{x_1})B'' + \mathcal{O}(\tilde{h}).$$

However,  $\chi(\tilde{h}D_{x_1})(hD_{x_1}) = \chi(\tilde{h}D_{x_1})(\tilde{h}D_{x_1})(h/\tilde{h}) = \mathcal{O}(h/\tilde{h})$ ; therefore,

$$A = \chi(\tilde{h}D_{x_1})\Psi_{\Pi}B' + \mathcal{O}(\tilde{h}).$$

We now put  $A_R = \chi(\tilde{h}D_{x_1})\Psi_{\Pi}B'$ . To estimate its rank, we write for  $\tilde{h}$  small enough,  $\chi(\tilde{h}D_{x_1})\Psi_{\Pi} = \chi(\tilde{h}D_{x_1}) \otimes \tilde{\Psi}_{\Pi}$ , where  $\chi(\tilde{h}D_{x_1})$  on the right-hand side acts on  $L^2(\mathbb{S}^1)$  and has rank  $\mathcal{O}(\tilde{h}^{-1})$ ; therefore, by (6.5),

$$\text{rank } A_R \leq C\tilde{h}^{\nu-n}h^{-\nu}$$

and the proof of Lemma 6.1 is finished.

## 7. PROOF OF THE MAIN LEMMA

Fix  $\delta > 0$  small enough so that all the results of §4 apply. We will now impose an additional condition on  $Q$ : namely, that there exists  $Q_0 \in \Psi^1(X)$  such that  $T_sQT_s^{-1} = \pm Q_0^*Q_0$  microlocally near  $\Sigma_{\pm}$ . Such a  $Q$  can be obtained by first choosing  $Q_0$  and  $T_s$ . Note that  $Q_0$  will be elliptic on  $\{(\xi)^{-2}p = 0\} \cap \{\mu \leq -\delta\}$ .

**7.1. Positive commutator estimates.** In §7.1 and §7.2 we assume  $|\text{Re } z| \leq C_0h$ ,  $|\text{Im } z| \leq C_0h$ . We relax this assumption in §7.3 where we prove an improved estimate for  $\text{Im } z > 0$ .

We start with the construction of an escape function near the trapped set. We recall [SjZw, Lemma 7.6] (where one puts  $\epsilon = h/\tilde{h}$ ):

**Lemma 7.1.** *Suppose the geodesic flow on  $M$  is hyperbolic on  $K$  in the sense of the assumption of Theorem 2. Then there exists a neighborhood  $V_K$  of  $\iota(K)$  and a family of smooth real-valued functions  $\hat{f}(x, \xi; h, \tilde{h})$  on  $V_K$  depending on two parameters  $0 < h < \tilde{h}$ , such that for some constant  $C_{\hat{f}} > 1$ ,*

$$\hat{f} = \mathcal{O}(\log(1/h)), \quad \partial^{\alpha} H_p^k \hat{f} = \mathcal{O}((\tilde{h}/h)^{-|\alpha|/2}), \quad |\alpha| + k \geq 1;$$

$$H_p \hat{f}(x, \xi) \geq C_{\hat{f}}^{-1} > 0 \text{ for } d((x, \xi), \iota(\tilde{K})) > C_{\hat{f}}(h/\tilde{h})^{1/2}.$$

Note that we have written  $\mathcal{O}(\log(1/h))$  where [SjZw] has  $\mathcal{O}(\log(\tilde{h}/h))$ . This is equivalent because when  $h < \tilde{h}^2 < 1$ ,  $\log(1/h)/2 \leq \log(\tilde{h}/h) \leq \log(1/h)$ . For later convenience, we assume that  $V_K$  is small enough that

$$\bar{V}_K \subset \{\mu > \sqrt{5\delta}\}, \tag{7.1}$$

and that the estimate (4.5) holds on  $\overline{V}_K$ .

Take a neighborhood  $U_K$  of  $K$  such that  $\overline{U}_K \subset V_K$ , and which is sufficiently small that Lemma 4.7 applies, and let  $f_0$  be the function defined by Lemma 4.7. Then define

$$F_0 \in \Psi^{\text{comp}}(X), \quad F_0^* = F_0, \quad \sigma(F_0) = f_0 + \mathcal{O}(h).$$

Here  $F_0$  is a standard compactly microlocalized semiclassical pseudodifferential operator in the sense of §3.1. Next take  $\hat{\chi} \in C_0^\infty(V_K)$  such that  $\hat{\chi} = 1$  near  $\overline{U}_K$ , and define

$$\hat{F} \in \log(1/h)\Psi_{1/2}^{\text{comp}}(X), \quad \text{WF}_h(\hat{F}) \subset V_K, \quad \hat{F}^* = \hat{F}, \quad \tilde{\sigma}(\hat{F}) = \hat{\chi}\hat{f} + \mathcal{O}(h^{1/2}\tilde{h}^{1/2}).$$

Here  $\hat{F}$  is a pseudodifferential operator in the exotic class described in Lemma 5.4. Let  $M$  be a large constant and define the full quantized escape function

$$F = \hat{F} + M \log(1/h)F_0.$$

Then  $F = \mathcal{O}(\log(1/h))\Psi_{1/2}^{\text{comp}}$  and its principal symbol is

$$f = \hat{\chi}\hat{f} + M \log(1/h)f_0.$$

Note that

$$f = \mathcal{O}(\log(1/h)), \quad \partial^\alpha f = \mathcal{O}((h/\tilde{h})^{-|\alpha|/2}), \quad |\alpha| > 0.$$

We then calculate

$$H_p f = \hat{\chi} \cdot H_p \hat{f} + \hat{f} \cdot H_p \hat{\chi} + M \log(1/h) H_p f_0; \quad (7.2)$$

we see by Lemma 7.1 that  $H_p f = \mathcal{O}(\log(1/h))\Psi_{1/2}^{\text{comp}}$  and therefore (see Lemma 5.2(7) and Lemma 5.4(3)) we gain a full power of  $h$  when commuting  $F$  with  $P(0)$ :

$$[P(0), F] = \mathcal{O}(h \log(1/h))\Psi_{1/2}^{\text{comp}}, \quad \tilde{\sigma}([P(0), F]) = -ih H_p f + \mathcal{O}(h\tilde{h})\Psi_{1/2}^{\text{comp}}. \quad (7.3)$$

Since  $\pm H_p f_0 > 0$  near the set  $V_\pm$  defined in Lemma 4.7(3), we can fix  $M$  large enough so that for some constant  $C_f > 0$ ,

$$\pm H_p f > C_f^{-1} \log(1/h) \text{ near } V_\pm. \quad (7.4)$$

We now perform the conjugation. Since  $F$  is compactly microlocalized and  $\|F\| = \mathcal{O}(\log(1/h))$ , we see that for any fixed  $t$ , the operator  $e^{tF} - 1$  is compactly microlocalized and polynomially bounded in  $h$ . Given the operator  $T_s$  from Lemma 4.6 (and in particular  $s$  is large enough depending on  $\text{Im } z$ ), define the conjugated operator

$$P_t(z) = e^{-tF} T_s(P(z) - iQ) T_s^{-1} e^{tF}.$$

Here  $t > 0$  will be chosen in Lemma 7.9 so that the contribution from the conjugation by  $e^{tF}$  is greater than that of  $\text{Im } T_s P(z) T_s^{-1}$ ; how large  $t$  needs to be depends on  $\text{Im } z$ .

**Lemma 7.2.** *We have*

$$P_t(z) = T_s(P(z) - iQ) T_s^{-1} + t[T_s(P(z) - iQ) T_s^{-1}, F] + \mathcal{O}_t(h\tilde{h})\Psi_{1/2}^{\text{comp}}. \quad (7.5)$$

*Proof.* We follow the proof of [SjZw, Proposition 8.2]. We write

$$\begin{aligned} P_t(z) &= e^{-t \operatorname{ad}_F} (T_s(P(z) - iQ)T_s^{-1}) = T_s(P(z) - iQ)T_s^{-1} - t \operatorname{ad}_F(T_s(P(z) - iQ)T_s^{-1}) \\ &\quad + \int_0^t (t - t') e^{-t'F} \operatorname{ad}_F^2(T_s(P(z) - iQ)T_s^{-1}) e^{t'F} dt'. \end{aligned} \quad (7.6)$$

By the Bony–Chemin Theorem ([BoCh]; see [SjZw, Lemma 8.1] for this version), the conjugation  $A \mapsto e^{-t'F} A e^{t'F}$  is continuous  $\Psi_{1/2}^{\operatorname{comp}} \rightarrow \Psi_{1/2}^{\operatorname{comp}}$ . Hence, it remains to show that

$$\operatorname{ad}_F^2(T_s(P(z) - iQ)T_s^{-1}) = \mathcal{O}(h\tilde{h})_{\Psi_{1/2}^{\operatorname{comp}}}.$$

Using Lemma 4.1(6) and the fact that  $|z| \leq 2C_0h$  (see the remark at the beginning of the subsection),  $\operatorname{WF}_h(\widehat{F}) \cap \operatorname{WF}_h(Q) = \emptyset$ , and Lemma 5.4(3),

$$[T_s(P(z) - iQ)T_s^{-1}, F] = [P(0), F] - iM \log(1/h)[Q, F_0] + \mathcal{O}(h^{3/2}\tilde{h}^{1/2})_{\Psi_{1/2}^{\operatorname{comp}}}. \quad (7.7)$$

The second term on the right-hand side is  $\mathcal{O}(h \log(1/h))_{\Psi^{\operatorname{comp}}}$ ; commuting it with  $F$ , we get  $\mathcal{O}(h^{3/2}\tilde{h}^{1/2} \log(1/h))$ . The third term is handled similarly. As for the first term, by (7.3) the symbol of  $h^{-1}[P(0), F]$  satisfies (5.8); therefore, by Lemma 5.4(2),  $[[P(0), F], F] \in h\tilde{h}\Psi_{1/2}^{\operatorname{comp}}$ .  $\square$

Combining Lemma 4.1(6),  $|z| \leq 2C_0h$ , (7.3), (7.5), and (7.7), we get

$$P_t(z) = P(0) - iQ + \mathcal{O}_t(h)_{\Psi^1} + \mathcal{O}_t(h \log(1/h))_{\Psi_{1/2}^{\operatorname{comp}}}. \quad (7.8)$$

Since  $q$  and  $f_0$  are real-valued,  $\operatorname{Re}[Q, F_0] = \mathcal{O}(h^2)_{\Psi^{\operatorname{comp}}}$ ; therefore, using (7.5) and (7.7),

$$\operatorname{Im} P_t(z) = \operatorname{Im}(T_s P(z) T_s^{-1}) - \operatorname{Re}(T_s Q T_s^{-1}) + t \operatorname{Im}[P(0), F] + \mathcal{O}_t(h\tilde{h})_{\Psi_{1/2}^{\operatorname{comp}}}. \quad (7.9)$$

We will use a positive commutator argument for  $\operatorname{Im} \tilde{P}_t(z)$ , with  $\tilde{P}_t(z) = P_t(z) - ithA$  as in (2.1), to control the norm of  $u$  in terms of  $\tilde{P}_t(z)u$ . We first analyze the terms involving  $P_t(z)$ , then define  $A$  and analyze the  $-ithA$  term, and finally put them all together in §§7.2–7.3. The right-hand side of (7.9) consists of several components in different symbol classes, with positivity of the sum provided by different components in different regions of  $\bar{T}^*X$ . In Lemma 7.4 we construct a microlocal partition of unity corresponding to these regions, and treat each member of the partition in a separate lemma. The following lemma is useful in dealing with this partition; the left-hand side of (7.10) is easily summed over different operators  $\Psi_1$ , and the right-hand side is adapted to a positive commutator argument.

**Lemma 7.3.** *Let  $\Psi_1 \in \Psi^0$  have real-valued principal symbol and assume that  $T_s Q T_s^{-1} = \pm Q_0^* Q_0$  near  $\operatorname{WF}_h(\Psi_1)$ . Then for  $u \in C^\infty(X)$ ,*

$$\begin{aligned} \pm \operatorname{Re} \langle \operatorname{Im} P_t(z)u, \Psi_1^2 u \rangle &\leq \pm \langle (\operatorname{Im}(T_s P(z) T_s^{-1})) \Psi_1 u, \Psi_1 u \rangle \\ &\mp t \operatorname{Re} \langle i[P(0), F] \Psi_1 u, \Psi_1 u \rangle + \mathcal{O}_t(h\tilde{h}) \|u\|_{L^2}^2. \end{aligned} \quad (7.10)$$



*Proof.* We first claim that the left-hand side can be replaced by  $\pm \langle (\operatorname{Im} P_t(z)) \Psi_1 u, \Psi_1 u \rangle$ :

$$\operatorname{Re} \langle \operatorname{Im} P_t(z) u, \Psi_1^2 u \rangle - \langle \operatorname{Im} P_t(z) \Psi_1 u, \Psi_1 u \rangle = \mathcal{O}_t(h^{3/2} \tilde{h}^{1/2} \log(1/h)) \|u\|_{L^2}. \quad (7.11)$$

Indeed, write the left-hand side as  $\operatorname{Re}(Bu, u)$ , where

$$B = \Psi_1^*((\Psi_1^* - \Psi_1) \operatorname{Im} P_t(z) + [\Psi_1, \operatorname{Im} P_t(z)]).$$

We now use (7.8). The part of  $B$  corresponding to  $P(0) - iQ + \mathcal{O}_t(h)_{\Psi_1}$  lies in  $h\Psi^1$  for each  $t$ , and, when multiplied by  $h^{-1}$ , has imaginary-valued principal symbol; therefore, the corresponding part of  $\operatorname{Re}(Bu, u)$  is  $\mathcal{O}_t(h^2) \|u\|_{L^2}^2$ . The part of  $B$  corresponding to  $\mathcal{O}_t(h \log(1/h))_{\Psi_{1/2}^{\text{comp}}}$  is  $\mathcal{O}_t(h^{3/2} \tilde{h}^{1/2} \log(1/h))_{\Psi_{1/2}^{\text{comp}}}$  by Lemma 5.2(6).

Having established (7.11), we use (7.9):

$$\begin{aligned} \pm \langle \operatorname{Im} P_t(z) \Psi_1 u, \Psi_1 u \rangle &= \pm \langle \operatorname{Im}(T_s P(z) T_s^{-1}) \Psi_1 u, \Psi_1 u \rangle \\ &\quad - \|Q_0 \Psi_1 u\|_{L^2}^2 \pm t \operatorname{Im} \langle [P(0), F] \Psi_1 u, \Psi_1 u \rangle + \mathcal{O}_t(h \tilde{h}) \|u\|_{L^2}^2; \end{aligned}$$

it remains to note that the second term on the right-hand side is  $\leq 0$ .  $\square$

We now introduce the microlocal partition of unity mentioned before Lemma 7.3. The operator  $\Psi_E$  corresponds to the elliptic set of  $P(z) - iQ$ ,  $\Psi_{L_{\pm}}$  correspond to the neighborhoods of the radial sets  $L_{\pm}$  where  $\operatorname{Im}(T_s P(z) T_s^{-1})$  has a favorable sign by Lemma 4.6,  $\Psi_{0\pm}$  correspond to the transition region, where the escape function  $f_0$  from Lemma 4.7 provides positivity, and  $\Psi_K$  handles a neighborhood of the trapped set.

**Lemma 7.4.** *There exist operators  $\Psi_E, \Psi_{L_{\pm}} \in \Psi^0(X)$  and  $\Psi_{0\pm}, \Psi_K \in \Psi^{\text{comp}}(X)$  with real-valued principal symbols, and neighborhoods  $\tilde{\Sigma}_{\pm}$  of  $\Sigma_{\pm} \cap \{\mu \geq -\delta\}$ , such that*

- (1)  $\operatorname{WF}_h(\Psi_E)$  is contained in the elliptic set of  $p - iq$ ;
- (2)  $\Psi_{L_+}^2 + \Psi_{0+}^2 + \Psi_K^2 = 1$  and  $\Psi_{L_-} = \Psi_{0-} = 0$  microlocally on  $\tilde{\Sigma}_+$  and  $\Psi_{L_-}^2 + \Psi_{0-}^2 = 1$  and  $\Psi_{L_+} = \Psi_{0+} = \Psi_K = 0$  microlocally on  $\tilde{\Sigma}_-$ ;
- (3)  $\Psi_E$  is elliptic on the complement of  $\tilde{\Sigma}_+ \cup \tilde{\Sigma}_-$ ;
- (4)  $T_s Q T_s^{-1} = \pm Q_0^* Q_0$  microlocally near  $\operatorname{WF}_h(\Psi_{L_{\pm}}) \cup \operatorname{WF}_h(\Psi_{0\pm})$ , and  $\operatorname{WF}_h(T_s Q T_s^{-1}) \cap \operatorname{WF}_h(\Psi_K) = \emptyset$ ;
- (5)  $\pm \langle \xi \rangle^{-1} \sigma(h^{-1} \operatorname{Im}(T_s P(z) T_s^{-1})) < 0$  and  $\pm H_p f_0 \geq 0$  near  $\operatorname{WF}_h(\Psi_{L_{\pm}})$ ; moreover,  $\operatorname{WF}_h(\Psi_{L_{\pm}}) \cap \bar{U}_K = \operatorname{WF}_h(\Psi_{L_{\pm}}) \cap \operatorname{WF}_h(\hat{F}) = \emptyset$ ;
- (6)  $\pm H_p f \geq C_f^{-1} \log(1/h) > 0$  on  $\operatorname{WF}_h(\Psi_{0\pm})$ ;
- (7)  $\operatorname{WF}_h(\Psi_{L_{\pm}})$  and  $\operatorname{WF}_h(\Psi_{0\pm})$  do not intersect  $\iota(\tilde{K})$ ;
- (8)  $\operatorname{WF}_h(\Psi_K) \subset U_K$  and  $H_p f_0 \geq 0$  near  $\operatorname{WF}_h(\Psi_K)$ ;
- (9)  $\mp \operatorname{Re}(\langle \xi \rangle^{-1} \sigma(\partial_z P(0))) > 0$  on  $\tilde{\Sigma}_{\pm}$ .

*Proof.* We will define open coverings

$$\Sigma_+ \cap \{\mu \geq -\delta\} \subset U_{L_+} \cup U_{0+} \cup U_K, \quad \Sigma_- \cap \{\mu \geq -\delta\} \subset U_{L_-} \cup U_{0-}, \quad (7.12)$$

take partitions of unity subordinate to these open coverings such that

$$\psi_{L_+}^2 + \psi_{0+}^2 + \psi_K^2 = 1 \text{ near } \Sigma_+ \cap \{\mu \geq -\delta\}, \quad \psi_{L_-}^2 + \psi_{0-}^2 = 1 \text{ near } \Sigma_- \cap \{\mu \geq -\delta\}, \quad (7.13)$$

satisfying the additional support conditions (possible since  $\Sigma_+ \cap \Sigma_- = \emptyset$ )

$$\begin{aligned} (\text{supp } \psi_{L_+} \cup \text{supp } \psi_{0+} \cup \text{supp } \psi_K) \cap (\Sigma_- \cap \{\mu \geq -\delta\}) &= \emptyset, \\ (\text{supp } \psi_{L_-} \cup \text{supp } \psi_{0-}) \cap (\Sigma_+ \cap \{\mu \geq -\delta\}) &= \emptyset, \end{aligned} \quad (7.14)$$

and take open  $\tilde{\Sigma}_\pm \supset \Sigma_\pm \cap \{\mu \geq -\delta\}$  such that (7.13) and (7.14) hold with  $\Sigma_\pm \cap \{\mu \geq -\delta\}$  replaced by  $\tilde{\Sigma}_\pm$ . The  $\Psi_j$  will be obtained at the end of this proof by quantizing the  $\psi_j$  and adding correction terms (without changing the semiclassical wavefront sets) to obtain the equations in item (2) without remainders. Since  $p - iq$  is elliptic on the complement of  $\tilde{\Sigma}_+ \cup \tilde{\Sigma}_-$  (as follows from the properties of  $q$  listed at the end of Section 4.1), we can find  $\Psi_E \in \Psi^0(X)$  such that items (1) and (3) hold.

The open set  $U_K$  is the same as the one defined immediately following Lemma 7.1. Since  $\text{WF}_h(\Psi_K) \subset U_K$ , the properties of  $\text{WF}_h(\Psi_K)$  asserted in items (4) and (8) follow from (7.1), which implies  $\overline{U}_K \subset \{\mu > 0\} \subset \overline{T}^*X \setminus \text{WF}_h(T_s Q T_s^{-1})$ , and from Lemma 4.7(2).

Let  $U_{L\pm}$  be open sets such that, with  $\rho_1$  defined in (4.10),

$$\Sigma_\pm \cap \{\mu \geq -\delta\} \cap \{\rho_1 \leq 5\delta\} \subset U_{L\pm}, \quad (7.15)$$

and such that  $\overline{U}_{L\pm}$  is disjoint from

$$\begin{aligned} &\text{WF}_h(T_s Q T_s^{-1} \mp Q_0^* Q_0) \cup \{\pm \langle \xi \rangle^{-1} \sigma(h^{-1} \text{Im}(T_s P(z) T_s^{-1})) \geq 0\} \cup \overline{U}_K \\ &\cup \text{WF}_h(\widehat{F}) \cup \overline{\{\pm H_p f_0 < 0\}} \cup \iota(\tilde{K}) \cup \{\mp \text{Re}(\langle \xi \rangle^{-1} \sigma(\partial_z P(0))) \leq 0\}. \end{aligned}$$

To see that such sets exist, note that  $\Sigma_\pm \cap \{\mu \geq -\delta\} \cap \{\rho_1 \leq 5\delta\}$  is disjoint from  $\text{WF}_h(T_s Q T_s^{-1} \mp Q_0^* Q_0)$  by the condition imposed on  $Q_0$  when it was introduced at the beginning of §7, from  $\{\pm \langle \xi \rangle^{-1} \sigma(h^{-1} \text{Im}(T_s P(z) T_s^{-1})) \geq 0\}$  by Lemma 4.6, from  $\overline{U}_K \cup \text{WF}_h(\widehat{F})$  by (7.1), from  $\{\pm H_p f_0 < 0\}$  by Lemma 4.7(2), from  $\iota(\tilde{K})$  by the fact that  $L_\pm$  is a source/sink (see (4.11)), and from  $\{\mp \text{Re}(\langle \xi \rangle^{-1} \sigma(\partial_z P(0))) \leq 0\}$  by (4.5). This disjointness condition implies the properties of  $\text{WF}_h(\Psi_{L\pm})$  asserted in items (4), (5), and (7).

Let  $U_{0\pm}$  be open sets such that

$$V_\pm \subset U_{0\pm}, \quad (7.16)$$

(with notation as in Lemma 4.7(3)) and such that  $\overline{U}_{0\pm}$  is disjoint from

$$\begin{aligned} &\text{WF}_h(T_s Q T_s^{-1} \mp Q_0^* Q_0) \cup \{\pm H_p f \leq C_f^{-1} \log(1/h)\} \\ &\cup \iota(\tilde{K}) \cup S^*X \cup \{\mp \text{Re}(\langle \xi \rangle^{-1} \sigma(\partial_z P(0))) \leq 0\}. \end{aligned}$$

That this is possible is checked as in the construction of  $U_{L\pm}$  above, and by (7.4). This disjointness condition implies the properties of  $\text{WF}_h(\Psi_{0\pm})$  asserted in items (4), (6), and (7). The condition of disjointness from  $S^*X$  ensures that  $\Psi_{0\pm}$  is compactly microlocalized.

The covering property (7.12) follows from (7.15) and (7.16). Now item (9) follows from

$$\tilde{\Sigma}_+ \subset U_{L_+} \cup U_{0+} \cup U_K, \quad \tilde{\Sigma}_- \subset U_{L_-} \cup U_{0-},$$

together with the fact that the closures of the right hand sides of this formula are disjoint from  $\{\mp \operatorname{Re}(\langle \xi \rangle^{-1} \sigma(\partial_z P(0))) \leq 0\}$  by construction.

We now explain in detail the construction of the  $\Psi_j$ , giving item (2). We have

$$\operatorname{Op}_h(\psi_{L_+})^2 + \operatorname{Op}_h(\psi_{0+})^2 + \operatorname{Op}_h(\psi_K)^2 = 1 + R_+$$

microlocally on  $\tilde{\Sigma}_+$ , where  $R_+ \in h\Psi^{-1}$ . There exists an operator  $S_+ = 1 + \mathcal{O}(h)_{\Psi^{-1}}$  such that  $S_+^2(1 + R_+) = 1$  microlocally on  $\tilde{\Sigma}_+$ . Then

$$(S_+ \operatorname{Op}_h(\psi_{L_+}))^2 + (S_+ \operatorname{Op}_h(\psi_{0+}))^2 + (S_+ \operatorname{Op}_h(\psi_K))^2 = 1 + \mathcal{O}(h^2)_{\Psi^{-2}},$$

microlocally on  $\tilde{\Sigma}_+$ . Iterating the process of dividing the right hand side over, and concluding with a Borel summation, we improve the remainder to  $\mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ , while preserving the property  $\operatorname{WF}_h(\Psi_j) \subset \operatorname{supp} \psi_j$ . The operators  $\Psi_{L_-}$  and  $\Psi_{0-}$  are constructed similarly.  $\square$

We now prove portions of the estimate corresponding to each of the pseudodifferential operators of Lemma 7.4. We start with the radial points, where we use the conjugation by  $T_s$ :

**Lemma 7.5.** *For some constant  $C_t$  and  $u \in C^\infty(X)$ ,*

$$\pm \operatorname{Re} \langle \operatorname{Im} P_t(z)u, \Psi_{L_\pm}^2 u \rangle \leq -C_t^{-1} h \|\Psi_{L_\pm} u\|_{H_h^{1/2}}^2 + \mathcal{O}_t(h\tilde{h}) \|u\|_{L^2}^2.$$

*Proof.* Note that, by Lemma 7.4(5),  $\operatorname{WF}_h(\hat{F}) \cap \operatorname{WF}_h(\Psi_{L_\pm}) = \emptyset$ . By Lemma 7.3, it is then enough to estimate

$$\pm \langle (\operatorname{Im}(T_s P(z) T_s^{-1})) \Psi_{L_\pm} u, \Psi_{L_\pm} u \rangle \mp t M \log(1/h) \operatorname{Re} \langle i[P(0), F_0] \Psi_{L_\pm} u, \Psi_{L_\pm} u \rangle.$$

We now use Lemma 7.4(5) again. By the non-sharp Gårding inequality (3.4), the first term is  $\leq -C_t^{-1} h \|\Psi_{L_\pm} u\|_{H_h^{1/2}}^2 + \mathcal{O}(h^\infty) \|u\|_{L^2}^2$ . Also, the principal symbol of  $\mp h^{-1} i[P(0), F_0]$  is equal to  $\mp H_p f_0 \leq 0$  near  $\operatorname{WF}_h(\Psi_{L_\pm})$ ; then the second term is  $\leq \mathcal{O}(h^2 \log(1/h)) \|u\|_{L^2}^2$  by the sharp Gårding inequality (3.5).  $\square$

Next, we deal with the transition region:

**Lemma 7.6.** *For some constant  $C_t$  and  $u \in C^\infty(X)$ ,*

$$\pm \operatorname{Re} \langle \operatorname{Im} P_t(z)u, \Psi_{0\pm}^2 u \rangle \leq -C_t^{-1} h \log(1/h) \|\Psi_{0\pm} u\|_{L^2}^2 + \mathcal{O}_t(h\tilde{h}) \|u\|_{L^2}^2.$$

*Proof.* By Lemma 7.3, it is enough to estimate

$$\pm \langle (\operatorname{Im}(T_s P(z) T_s^{-1})) \Psi_{0\pm} u, \Psi_{0\pm} u \rangle \mp t \operatorname{Re} \langle i[P(0), F] \Psi_{0\pm} u, \Psi_{0\pm} u \rangle.$$

Since  $\operatorname{Im}(T_s P(z) T_s^{-1}) \in h\Psi^1$  and  $\Psi_{0\pm}$  is compactly microlocalized, the first term is  $\mathcal{O}(h) \|\Psi_{0\pm} u\|_{L^2}^2$ . Therefore, it is enough to show that

$$\mp \operatorname{Re} \langle i[P(0), F] \Psi_{0\pm} u, \Psi_{0\pm} u \rangle \leq -C^{-1} h \log(1/h) \|\Psi_{0\pm} u\|_{L^2}^2 + \mathcal{O}(h^\infty) \|u\|_{L^2}^2;$$

by (7.3) and Lemma 7.4(6), this follows by Lemma 5.3 applied to  $(h \log(1/h))^{-1}i[P(0), F]$ .  $\square$

To deal with the  $\Psi_K$  term, we have to modify our operator, adding a term  $-ith\tilde{A}$ , which provides positivity in an  $\mathcal{O}((h/\tilde{h})^{1/2})$  size neighborhood of the trapped set. Note that the resulting operator is not yet  $\tilde{P}_t(z)$ ; the final operator  $A$  that we use will also have a second microlocalization factor, introduced below. Let  $C_{\tilde{f}}$  be the constant from Lemma 7.1. Let  $\chi_1 \in C^\infty(\mathbb{R})$  be a nonnegative function such that  $\chi_1(\lambda) + \lambda = 1$  for  $\lambda \leq C_{\tilde{f}}^{-1}/2$  and  $\text{supp } \chi_1 \subset (-\infty, C_{\tilde{f}}^{-1})$ . Then the function  $\chi_1(H_p \hat{f})$ , defined on  $V_K$ , is supported  $\mathcal{O}((h/\tilde{h})^{1/2})$  close to  $\iota(\tilde{K})$ . Moreover,  $\chi_1(H_p \hat{f}) > 0$  in the region where  $H_p \hat{f}$  is not positive and

$$\chi_1(H_p \hat{f}) + H_p \hat{f} \geq C_{\tilde{f}}^{-1}/2 > 0 \text{ on } V_K. \quad (7.17)$$

Take a real-valued  $\tilde{\chi} \in C_0^\infty(U_K)$  equal to 1 near  $\text{WF}_h(\Psi_K)$ . Then the function

$$\tilde{a} = \chi_1(H_p \hat{f})\tilde{\chi} \quad (7.18)$$

is in  $C_0^\infty(U_K)$  and thus can be extended to  $\bar{T}^*X$ . It follows from Lemma 7.1 that  $\tilde{a}$  lies in the exotic class  $S_{1/2}^{\text{comp}}$  from §5.1:

$$\partial_{x,\xi}^\alpha H_p^k \tilde{a} = \mathcal{O}((h/\tilde{h})^{-|\alpha|/2}). \quad (7.19)$$

Let  $\tilde{A} \in \Psi_{1/2}^{\text{comp}}$  be any self-adjoint quantization of  $\tilde{a}$ ; note that  $\text{WF}_h(\tilde{A}) \subset U_K$ . In fact,  $\text{WF}_h(\tilde{A}) \subset \text{supp } \tilde{\chi} \cap \iota(\tilde{K})$ , because by Lemma 7.1, for a fixed  $(x, \xi) \notin \iota(\tilde{K})$  and  $h/\tilde{h}$  small enough, we have  $H_p \hat{f}(x, \xi) \geq C_{\tilde{f}}^{-1}$  and thus  $\chi_1(H_p \hat{f}(x, \xi)) = 0$ . In particular, recalling Lemma 4.7(4), we have

$$\text{WF}_h(\tilde{A}) \cap \text{supp}(H_p f_0) = \emptyset. \quad (7.20)$$

**Lemma 7.7.** *For  $t$  large enough, some constant  $C_t$  and  $u \in C^\infty(X)$ ,*

$$\text{Re}\langle \text{Im}(P_t(z) - ith\tilde{A})u, \Psi_K^2 u \rangle \leq -C_t^{-1}h\|\Psi_K u\|_{L^2}^2 + \mathcal{O}_t(h\tilde{h})\|u\|_{L^2}^2.$$

*Proof.* By Lemma 7.3, and using that  $[\tilde{A}, \Psi_K] = \mathcal{O}(h^{1/2}\tilde{h}^{1/2})_{\Psi_{1/2}^{\text{comp}}}$  by Lemma 5.2(6), we see that it suffices to estimate

$$\begin{aligned} & \langle (\text{Im}(T_s P(z) T_s^{-1})) \Psi_K u, \Psi_K u \rangle - Mt \log(1/h) \text{Re}\langle i[P(0), F_0] \Psi_K u, \Psi_K u \rangle \\ & - t \text{Re}\langle i[P(0), \hat{F}] \Psi_K u, \Psi_K u \rangle - th \langle \tilde{A} \Psi_K u, \Psi_K u \rangle. \end{aligned}$$

Since  $\text{Im}(T_s P(z) T_s^{-1}) \in h\Psi^1$  and  $\Psi_K$  is compactly microlocalized, the first term can be estimated by  $C_1 h \|\Psi_K u\|_{L^2}^2$ , where  $C_1$  is independent of  $t$ . Since  $\sigma(h^{-1}i[P(0), F_0]) = H_p f_0 \geq 0$  near  $\text{WF}_h(\Psi_K)$  by Lemma 7.4(8), the second term is  $\leq \mathcal{O}_t(h^2 \log(1/h))\|u\|_{L^2}^2$  by sharp Gårding inequality (3.5). Therefore, it suffices to pick  $t$  large enough and prove that for some constant  $C_2$  independent of  $t$ , we have

$$- \text{Re}(\langle i[P(0), \hat{F}] + h\tilde{A} \rangle \Psi_K u, \Psi_K u) \leq -C_2^{-1}h\|\Psi_K u\|_{L^2}^2 + \mathcal{O}(h^\infty)\|u\|_{L^2}^2. \quad (7.21)$$

Near  $\text{WF}_h(\Psi_K) \subset U_K$ ,  $\hat{\chi} = 1$  and  $H_p(\hat{\chi}\hat{f}) = H_p\hat{f} = \mathcal{O}(1)_{S_{1/2}^{\text{comp}}}$ ; therefore, by Lemma 5.4(3),  $i[P(0), \hat{F}] + h\tilde{A} \in h\Psi_{1/2}^{\text{comp}}$  and, when multiplied by  $h^{-1}$ , its principal symbol is  $H_p\hat{f} + \chi_1(H_p\hat{f}) + \mathcal{O}(\tilde{h})$ ; by (7.17) this symbol is positive and it remains to apply Lemma 5.3.  $\square$

We will now fix  $t$  and forget the dependence of the remainders on it.

It is finally time to use second microlocalization and construct the operator  $A$ . Let  $\hat{P} \in \Psi^2$  be any self-adjoint operator elliptic near the fiber infinity  $S^*X$  and whose principal symbol  $\hat{p}$  is equal to  $p$  in  $U_K$ . Take a function  $\chi \in C_0^\infty(\mathbb{R})$  equal to 1 near 0 and put

$$A = \chi((\tilde{h}/h)\hat{P})\tilde{A}, \quad \tilde{P}_t(z) = P_t(z) - ithA. \quad (7.22)$$

We use the ellipticity of  $\tilde{P}_t(z)$  away from the energy surface to estimate the difference  $A - \tilde{A}$ :

**Lemma 7.8.** *For  $u \in C^\infty(X)$ , and any  $N$*

$$\|\text{Re}(A - \tilde{A})u\|_{L^2} \leq \mathcal{O}(\tilde{h}/h)\|\tilde{P}_t(z)u\|_{H_h^{-N}} + \mathcal{O}(\tilde{h})\|u\|_{L^2}.$$

*Proof.* Since both  $\tilde{A}$  and  $\chi((\tilde{h}/h)\hat{P})$  are self-adjoint, we get

$$\text{Re}(A - \tilde{A}) = \frac{1}{2}[\tilde{A}, \chi((\tilde{h}/h)\hat{P})] - (1 - \chi((\tilde{h}/h)\hat{P}))\tilde{A}.$$

Now, by (7.19) and Lemma 5.8(2),  $[\tilde{A}, \chi((\tilde{h}/h)\hat{P})] = \mathcal{O}(\tilde{h})$ , so we can drop the commutator term. Write  $\chi(\lambda) - 1 = \lambda\psi(\lambda)$ , where  $\psi$  is a bounded function. By the functional calculus,

$$-(1 - \chi((\tilde{h}/h)\hat{P}))\tilde{A} = \psi((\tilde{h}/h)\hat{P})(\tilde{h}/h)\hat{P}\tilde{A};$$

since  $\psi((\tilde{h}/h)\hat{P})$  is bounded on  $L^2$  uniformly in  $h, \tilde{h}$ , it is enough to prove that

$$\|\hat{P}\tilde{A}u\|_{L^2} \leq \mathcal{O}(1)\|\tilde{P}_t(z)u\|_{H_h^{-N}} + \mathcal{O}(h)\|u\|_{L^2}.$$

Since  $[\hat{P}, \tilde{A}] = \mathcal{O}(h)$  by (7.19) and Lemma 5.2(7), this reduces to

$$\|\tilde{A}\hat{P}u\|_{L^2} \leq \mathcal{O}(1)\|\tilde{P}_t(z)u\|_{H_h^{-N}} + \mathcal{O}(h)\|u\|_{L^2}.$$

Now,  $\hat{P} = T_s(P(0) - iQ)T_s^{-1} + \mathcal{O}(h)$  microlocally in  $U_K$ ; therefore, it suffices to show that

$$\|\tilde{A}T_s(P(0) - iQ)T_s^{-1}u\|_{L^2} \leq \mathcal{O}(1)\|\tilde{P}_t(z)u\|_{H_h^{-N}} + \mathcal{O}(h)\|u\|_{L^2}.$$

The latter estimate follows from (7.5),  $P_t(z) = \tilde{P}_t(z) + \mathcal{O}(h)$ , (7.7), and the fact that  $[P(0) - iQ, F] = \mathcal{O}(h)$  microlocally near  $\text{WF}_h(\tilde{A})$ ; indeed,  $Q = 0$  and  $[P(0), \hat{F}] = \mathcal{O}(h)$  microlocally on  $U_K \supset \text{WF}_h(\tilde{A})$ , and  $[P(0), F_0] = \mathcal{O}(h^2)$  microlocally near  $\text{WF}_h(\tilde{A})$  by (7.20).  $\square$

We can now combine the previous two lemmas to prove

**Lemma 7.9.** *For  $t$  large enough, some constant  $C_t$ , and  $u \in C^\infty(X)$ ,*

$$\text{Re}(\text{Im } \tilde{P}_t(z)u, \Psi_K^2 u) \leq -C_t^{-1}h\|\Psi_K u\|_{L^2}^2 + \mathcal{O}_t(h\tilde{h})\|u\|_{L^2}^2 + \mathcal{O}_t(\tilde{h})\|\tilde{P}_t(z)u\|_{H_h^{-N}}\|u\|_{L^2}.$$

*Proof.* The left-hand side is

$$\operatorname{Re}\langle \operatorname{Im}(P_t(z) - ith\tilde{A})u, \Psi_K^2 u \rangle - th \operatorname{Re}\langle \operatorname{Re}(A - \tilde{A})u, \Psi_K^2 u \rangle;$$

the first term is estimated by Lemma 7.7 and the second term is estimated by Lemma 7.8.  $\square$

**7.2. Proof of Lemma 2.2(1).** First, let  $\Psi_1 \in \{\Psi_{L_\pm}, \Psi_{0\pm}\}$ . Then

$$2\Psi_1^*(\operatorname{Re} A) = [\Psi_1^*, \chi((\tilde{h}/h)\hat{P})]\tilde{A} + \chi((\tilde{h}/h)\hat{P})\Psi_1^*\tilde{A} + \Psi_1^*\tilde{A}\chi((\tilde{h}/h)\hat{P}).$$

This operator is  $\mathcal{O}(\tilde{h})$ , as the commutator above is  $\mathcal{O}(\tilde{h})$  by Lemma 5.8(2) and  $\Psi_1^*\tilde{A} = \mathcal{O}(h^\infty)$  by Lemma 7.4(7). Hence Lemmas 7.5 and 7.6 are valid for  $\tilde{P}_t(z)$  in place of  $P_t(z)$ . Now, put

$$Z = \Psi_{L_+}^2 + \Psi_{0+}^2 + \Psi_K^2 - \Psi_{L_-}^2 - \Psi_{0-}^2; \quad (7.23)$$

then  $Z = \pm 1$  microlocally on  $\tilde{\Sigma}_\pm$  by Lemma 7.4(2). For  $u \in C^\infty(X)$ , we have

$$\operatorname{Im}\langle \tilde{P}_t(z)u, Zu \rangle = \langle \operatorname{Im} \tilde{P}_t(z)u, Zu \rangle + \frac{1}{2i} \langle u, ([\tilde{P}_t(z), Z] + (Z - Z^*)\tilde{P}_t(z))u \rangle.$$

However, by (7.8), Lemma 5.2(6), and the fact that the principal symbol of  $Z$  is real,

$$[P_t(z), Z] + (Z - Z^*)P_t(z) = \mathcal{O}(h)_{\Psi_1} + \mathcal{O}(h^{3/2}\tilde{h}^{1/2}\log(1/h))_{\Psi_{1/2}^{\operatorname{comp}}},$$

and it is microlocalized outside of  $\tilde{\Sigma}_+ \cup \tilde{\Sigma}_-$ , that is, on the elliptic set of  $\Psi_E$ . Also,  $[A, Z] + (Z - Z^*)A = \mathcal{O}(\tilde{h})_{L^2 \rightarrow L^2}$  by Lemma 5.8(2). Therefore, by Lemmas 7.5, 7.6, and 7.9, for  $t$  large enough, we have

$$\begin{aligned} \operatorname{Im}\langle \tilde{P}_t(z)u, Zu \rangle &= \langle \operatorname{Im} \tilde{P}_t(z)u, Zu \rangle + \mathcal{O}(h)\|\Psi_E u\|_{H_h^{1/2}}^2 + \mathcal{O}(h\tilde{h})\|u\|_{L^2}^2 \\ &\leq -C^{-1}h(\|\Psi_{L_+} u\|_{H_h^{1/2}}^2 + \|\Psi_{0+} u\|_{H_h^{1/2}}^2 + \|\Psi_K u\|_{H_h^{1/2}}^2 + \|\Psi_{L_-} u\|_{H_h^{1/2}}^2 + \|\Psi_{0-} u\|_{H_h^{1/2}}^2) \\ &\quad + \mathcal{O}(h\tilde{h})\|u\|_{L^2}^2 + \mathcal{O}(h)\|\Psi_E u\|_{H_h^{1/2}}^2 + \mathcal{O}(\tilde{h})\|\tilde{P}_t(z)u\|_{H_h^{-N}}\|u\|_{L^2}. \end{aligned} \quad (7.24)$$

Combining this with Lemma 7.4(3), we get

$$\|u\|_{H_h^{1/2}}^2 \leq \mathcal{O}(1)\|\Psi_E u\|_{H_h^{1/2}}^2 + \mathcal{O}(\tilde{h})\|u\|_{L^2}^2 + \mathcal{O}(h^{-1})\|\tilde{P}_t(z)u\|_{H_h^{-1/2}}\|u\|_{H_h^{1/2}}$$

and therefore for  $\tilde{h}$  small enough,

$$\|u\|_{H_h^{1/2}} \leq \mathcal{O}(1)\|\Psi_E u\|_{H_h^{1/2}} + \mathcal{O}(h^{-1})\|\tilde{P}_t(z)u\|_{H_h^{-1/2}}. \quad (7.25)$$

Finally, by Lemma 7.4(1) and the elliptic estimate (3.3),

$$\|\Psi_E u\|_{H_h^{1/2}} \leq \mathcal{O}(1)\|(P(0) - iQ)u\|_{H_h^{-1/2}} + \mathcal{O}(h^\infty)\|u\|_{H_h^{1/2}};$$

combining this with (7.8) and using  $\tilde{P}_t(z) = P_t(z) + \mathcal{O}(h)$ , we get

$$\|\Psi_E u\|_{H_h^{1/2}} \leq \mathcal{O}(1)\|\tilde{P}_t(z)u\|_{H_h^{-1/2}} + \mathcal{O}(h\log(1/h))\|u\|_{H_h^{1/2}}; \quad (7.26)$$

substituting this into (7.25) and removing the  $\mathcal{O}(h\log(1/h))$  error, we obtain (2.2).  $\square$

**7.3. Proof of Lemma 2.2(2).** We follow the proof of part (1), but with an additional positive term coming from  $\text{Im } z > 0$ . Let  $Z$  be as in (7.23); applying (7.24) to  $\text{Re } z$  instead of  $z$  and dropping the negative terms on the right hand side, we get

$$\text{Im} \langle \tilde{P}_t(\text{Re } z)u, Zu \rangle \leq \mathcal{O}(h\tilde{h})\|u\|_{L^2}^2 + \mathcal{O}(h)\|\Psi_E u\|_{H_h^{1/2}}^2 + \mathcal{O}(\tilde{h})\|\tilde{P}_t(\text{Re } z)u\|_{H_h^{-1/2}}\|u\|_{L^2}. \quad (7.27)$$

Now, by Lemma 4.1(6)

$$P(z) - P(\text{Re } z) = \mathcal{O}(|\text{Im } z|)_{\Psi^1},$$

and so

$$T_s(P(z) - P(\text{Re } z))T_s^{-1} = i \text{Im } z \partial_z P(0) + \mathcal{O}(|\text{Im } z|^2 + h|\text{Im } z|)_{\Psi^1}. \quad (7.28)$$

The conjugation by  $e^{tF}$  maps  $\Psi^1$  to  $\Psi^1 + \Psi_{1/2}^{\text{comp}}$  continuously, by the Bony–Chemin theorem (see the proof of Lemma 7.2). Moreover, we have by Lemma 5.4(3)

$$[F, \partial_z P(0)] = \mathcal{O}(h^{1/2}\tilde{h}^{1/2}|\text{Im } z|)_{\Psi_{1/2}^{\text{comp}}}. \quad (7.29)$$

Using  $\tilde{P}_t(z) - \tilde{P}_t(\text{Re } z) = P_t(z) - P_t(\text{Re } z)$ , the expansion (7.6), (7.28), (7.29) and proceeding as in the proof of Lemma 7.2, we get

$$\tilde{P}_t(z) - \tilde{P}_t(\text{Re } z) = i \text{Im } z \partial_z P(0) + \mathcal{O}(|\text{Im } z|^2 + h^{1/2}\tilde{h}^{1/2}|\text{Im } z|)_{\Psi^1 + \Psi_{1/2}^{\text{comp}}}.$$

Therefore, by (7.27) and since  $\text{Im } z \geq C_0 h$

$$\begin{aligned} \text{Im} \langle \tilde{P}_t(z)u, Zu \rangle &\leq \text{Im } z \text{Re} \langle \partial_z P(0)u, Zu \rangle + \mathcal{O}(h)\|\Psi_E u\|_{H_h^{1/2}}^2 \\ &\quad + \mathcal{O}(\tilde{h})\|\tilde{P}_t(z)u\|_{H_h^{-1/2}}\|u\|_{L^2} + \mathcal{O}(|\text{Im } z|^2 + \tilde{h}|\text{Im } z|)\|u\|_{H_h^{1/2}}^2. \end{aligned}$$

Now Lemma 7.4(3) and Lemma 7.4 (9) imply that  $\Psi_E$  is elliptic on  $\{\text{Re}(\langle \xi \rangle^{-1} \sigma(Z^* \partial_z P(0))) \geq 0\}$ , so by the non-sharp Gårding inequality (3.4) we get

$$\text{Re} \langle \partial_z P(0)u, Zu \rangle \leq -C^{-1}\|u\|_{H_h^{1/2}}^2 + C\|\Psi_E u\|_{H_h^{1/2}}^2.$$

Therefore, since  $C_0 h \leq \text{Im } z \leq \varepsilon$ ,

$$\|u\|_{H_h^{1/2}}^2 \leq C(\|\Psi_E u\|_{H_h^{1/2}}^2 + (\text{Im } z)^{-1}\|\tilde{P}_t(z)u\|_{H_h^{-1/2}}\|u\|_{H_h^{1/2}}) + \mathcal{O}(\tilde{h} + |\text{Im } z|)\|u\|_{H_h^{1/2}}^2.$$

Combining this with (7.26) and the fact that  $\tilde{h}$  and  $\text{Im } z$  are small, we get (2.3).  $\square$

**7.4. Proof of Lemma 2.2(3).** By Lemma 6.1 and (7.22), it suffices to show that

$$V_R = \{\exp(tH_{\hat{p}})(x, \xi) \mid |t| \leq R, (x, \xi) \in (\text{supp } \tilde{a} \cap \hat{p}^{-1}(0)) + B_{\hat{p}^{-1}(0)}(R(h/\tilde{h})^{1/2})\}$$

has, as a subset of  $\hat{p}^{-1}(0)$ ,  $2n - 1$  dimensional volume  $\mathcal{O}((h/\tilde{h})^{n-1-\nu})$ . Here  $2\nu + 1$  is bigger than the upper Minkowski dimension of  $K$ , or equal to it in the case of a trapped set of pure dimension. By the definition (7.18) of  $\tilde{a}$  and the fact that  $\text{supp } \chi_1 \subset (-\infty, C_{\hat{f}}^{-1})$  together with Lemma 7.1, we see that  $\text{supp } \tilde{a} \subset \iota(\tilde{K}) + B(C_{\hat{f}}(h/\tilde{h}))^{1/2}$ , so that

$$V_R \subset \{\exp(tH_{\hat{p}})(x, \xi) \mid |t| \leq R, (x, \xi) \in \hat{p}^{-1}(0) \cap (\iota(\tilde{K}) + B((R + C_{\hat{f}})(h/\tilde{h})^{1/2}))\}.$$



However, note that  $\iota(\tilde{K})$  is invariant under  $\exp(tH_{\tilde{p}})$ ; therefore, there exists a constant  $R'$  depending on  $R$  such that

$$V_R \subset \hat{p}^{-1}(0) \cap (\iota(\tilde{K}) + B(R'(h/\tilde{h})^{1/2})).$$

By (4.7) and since  $p_0 + 1$  is a homogeneous polynomial of degree 2 in the fibers,  $\tilde{K}$  is diffeomorphic to the product of  $K = \tilde{K} \cap p^{-1}(0)$  and an interval; therefore, for some  $R''$ ,

$$V_R \subset \iota(K) + B_{\hat{p}^{-1}(0)}(R''(h/\tilde{h})^{1/2}).$$

By the definition (1.3), the volume of an  $\varepsilon$ -neighborhood of  $\iota(K)$  is  $\mathcal{O}(\varepsilon^{2(n-1-\nu)})$ ; thus

$$\text{Vol}_{\hat{p}^{-1}(0)}(V_R) \leq C(h/\tilde{h})^{n-1-\nu}. \quad \square$$

## APPENDIX A. QUASIFUCHSIAN CONVEX COCOMPACT GROUPS

In this Appendix we describe in more detail the construction of the groups used in the examples in Figure 1. Recall that a finitely generated discrete group of Möbius transformations of the Riemann sphere  $\mathbb{C} \cup \{\infty\}$  is *Fuchsian* if it keeps invariant some disk or half-plane. Let  $\Gamma_0$  be the Fuchsian group generated by  $\{\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2\}$ , where all the transformations preserve the unit disk, and  $\mathcal{A}_1$  maps the exterior of the disk  $C_1$  onto the interior of the disk  $C_3$ ,  $\mathcal{B}_1$  maps the exterior of  $C_2$  onto the interior of  $C_4$ , and so on (see Figure 5). If  $\Gamma_0$  acts on the unit disk model of  $\mathbb{H}^2$ , then  $\Gamma_0 \backslash \mathbb{H}^2$  is a compact surface of genus 2 (see e.g. [Ka, §4.3, Example C]).

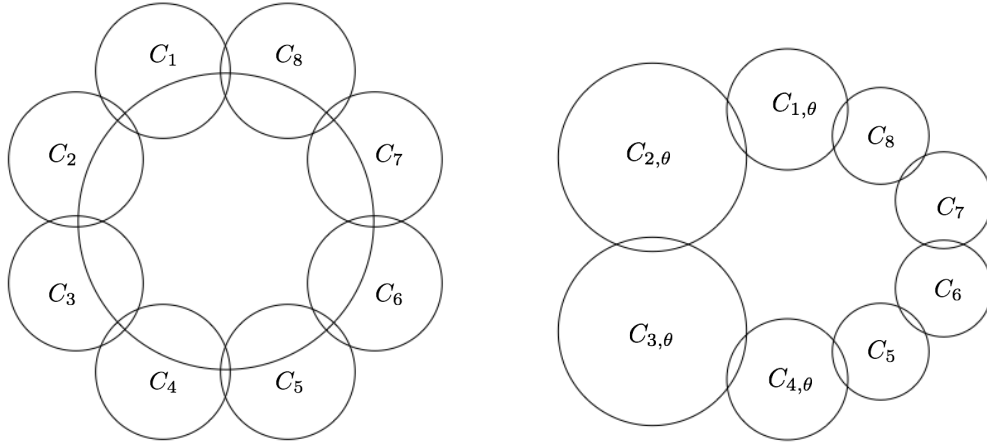


FIGURE 5. (Quasifuchsian bending.) The circles  $C_1, \dots, C_8$  are orthogonal to the unit circle and enclose a regular hyperbolic octagon of area  $4\pi$ . The circles  $C_{1,\theta}, \dots, C_{4,\theta}$  are the images of  $C_1, \dots, C_4$  under  $M_\theta$ .

If  $\Gamma_0$  instead acts on  $\mathbb{H}^3$  by Möbius transformations on the sphere at infinity ( $\mathcal{A}_1$  extends to an isometry of  $\mathbb{H}^3$  by mapping the half space whose boundary at infinity is the exterior of  $C_1$  to the half space whose boundary at infinity is the interior of  $C_3$ , and so on) then

$\Gamma_0 \backslash \mathbb{H}^3$  is isometric to  $\mathbb{R} \times \Gamma_0 \backslash \mathbb{H}^2$  with metric  $dr^2 + (\cosh^2 r)dS$ , where  $dS$  is the metric on  $\Gamma_0 \backslash \mathbb{H}^2$ , and  $\Gamma_0 \backslash \mathbb{H}^3$  is convex cocompact with limit set the unit circle, and  $\delta_{\Gamma_0} = 1$ .

Let  $\Gamma_\theta = \langle M_\theta \mathcal{A}_1 M_\theta^{-1}, M_\theta \mathcal{B}_1 M_\theta^{-1}, \mathcal{A}_2, \mathcal{B}_2 \rangle$ , where  $M_\theta$  is the rotation of the Riemann sphere  $\mathbb{C} \cup \{\infty\}$  by angle  $\theta$  which fixes  $\{i, -i\}$  and moves  $\mathbb{R}$  to the left (so that  $M_\theta \mathcal{A}_1 M_\theta^{-1}$  maps the exterior of  $C_{1,\theta}$  onto the interior of  $C_{3,\theta}$ , and so on). This is a *quasifuchsian bending* of  $\Gamma_0$  in the sense of [Ma, §VIII.E.3]. When  $\theta \neq 0$  the group is no longer Fuchsian because e.g.  $\text{Tr } M_\theta \mathcal{A}_1 M_\theta^{-1} \mathcal{A}_2 \notin \mathbb{R}$ , and hence  $\Gamma_\theta$  is not a group of isometries of  $\mathbb{H}^2$ . Nonetheless, it is still a group of isometries of  $\mathbb{H}^3$  and, for  $|\theta| \neq 0$  small enough,  $\Gamma_\theta \backslash \mathbb{H}^3$  is still convex cocompact and diffeomorphic to  $\mathbb{R} \times \Gamma_0 \backslash \mathbb{H}^2$ , but the metric is no longer a warped product and the limit set  $\Lambda_{\Gamma_\theta}$  is now a *quasicircle* with dimension  $\delta_{\Gamma_\theta} \in (1, 2)$  [Bo, Su84, BiJo]. In Figure 1 we plot  $\Lambda_{\Gamma_\theta}$  for  $\theta = 0.5$ , using Mathematica code based on that of [Ge, Appendix].

## APPENDIX B. THE HYPERBOLIC CYLINDER

In this Appendix we consider the hyperbolic cylinder  $M = (-1, 1)_r \times \mathbb{S}_y^1$  with metric

$$g = \frac{dr^2}{(1-r^2)^2} + \frac{d\tilde{y}^2}{1-r^2}.$$

We explain how this asymptotically hyperbolic manifold fits into the general framework of [Va11] and why Figure 2 represents the phase space picture for the modified operator.

Theorems 1 and 2 will apply with  $\nu = \delta_\Gamma = 0$ : note that  $M \simeq \langle z \mapsto e^{2\pi z} \rangle \backslash \mathbb{H}^2$ , where we use the upper half plane model of  $\mathbb{H}^2$ . In this case the resonances are actually known to lie on a lattice [GuZw95b, Appendix]. More generally, when as in this case the trapped set consists of a single hyperbolic orbit, the resonances are asymptotic to a lattice [GéSj].

First, note that we can bring the metric to the form (1.2) near  $\{r = \pm 1\}$  by taking

$$\tilde{x} = 2\sqrt{\frac{1 \mp r}{1 \pm r}}, \quad g = \frac{d\tilde{x}^2}{\tilde{x}^2} + \left(1 + \frac{\tilde{x}^2}{4}\right)^2 \frac{d\tilde{y}^2}{\tilde{x}^2}.$$

Then a boundary defining function of  $\overline{M}_{\text{even}}$  is given by

$$\mu = 1 - r^2.$$

(Strictly speaking, for the calculations in §4 and [Va11] to go through without changes, we need  $\mu = \tilde{x}^2$  near the conformal boundary; however, our  $\mu$  makes the formulas simpler and as  $\mu = \tilde{x}^2(1 + \mathcal{O}(\tilde{x}^2))$ , the analysis is the same.) The Laplacian on  $M$  is

$$\Delta_g = (1-r^2)^2 D_r^2 + ir(1-r^2)D_r + (1-r^2)D_y^2.$$

To simplify the formula for the modified Laplacian (4.1), we put  $e^\phi = \mu^{1/2}$ . We have

$$\begin{aligned} P(z) &= \mu^{-5/4} \mu^{i(z+1)/(2h)} (h^2(\Delta_g - 1/4) - (z+1)^2) \mu^{-i(z+1)/(2h)} \mu^{1/4} \\ &= \mu(hD_r)^2 + 2(z+1)r(hD_r) + D_y^2 - (z+1)^2 + \mathcal{O}(h)_{\Psi^1}. \end{aligned}$$

This operator extends to  $X = \mathbb{R}_r \times \mathbb{S}_y^1$  (in the rest of the paper, and in [Va11],  $X$  is compact, but we will not need this here). Note that for  $\mu > 0$  it is elliptic (Laplacian-like) but for  $\mu < 0$  it is hyperbolic (d'Alembertian-like). Take coordinates  $(r, \tilde{y}, \zeta, \tilde{\eta})$  on  $T^*X$ , with  $\zeta$  dual to  $r$  and  $\tilde{\eta}$  dual to  $\tilde{y}$ . We use the momentum  $\zeta$  instead of the momentum  $\tilde{\xi} = -\zeta/(2r)$ , dual to  $\mu$ , to avoid a coordinate singularity at  $r = 0$ . The principal symbol of  $P(0)$  is

$$p = \mu\zeta^2 + 2r\zeta + \tilde{\eta}^2 - 1, \quad (\text{B.1})$$

and the Hamiltonian flow is

$$H_p = 2(\mu\zeta + r)\partial_r + 2\zeta(r\zeta - 1)\partial_\zeta + 2\tilde{\eta}\partial_{\tilde{y}}. \quad (\text{B.2})$$

Our phase space  $\overline{T}^*X$  is as in §3.1 and is a ball bundle on  $X$ , and we denote a typical point by  $(x, \xi)$ . We now study the characteristic set  $\{\langle \xi \rangle^{-2}p = 0\}$  and the rescaled Hamiltonian flow  $\langle \xi \rangle^{-1}H_p$ , beginning with the behavior near fiber infinity (the prefactors  $\langle \xi \rangle^{-2}$  and  $\langle \xi \rangle^{-1}$  make the symbol and flow extend smoothly to fiber infinity,  $S^*X = \partial\overline{T}^*X$ ). We use the coordinates  $\check{\zeta} = \langle \xi \rangle^{-1}\zeta$ ,  $\check{\eta} = \langle \xi \rangle^{-1}\tilde{\eta}$  on the fibers in  $\overline{T}^*X$ ; then  $(\check{\zeta}, \check{\eta})$  lies on the circle of radius  $\langle \xi \rangle^{-1}|\xi|$  and

$$\langle \xi \rangle^{-2}p = \mu\check{\zeta}^2 + 2r\langle \xi \rangle^{-1}\check{\zeta} + \check{\eta}^2 - \langle \xi \rangle^{-2}.$$

On  $S^*X$ , we have  $\langle \xi \rangle^{-1} = 0$  and thus the characteristic set is given by

$$\{\langle \xi \rangle^{-2}p = 0\} \cap S^*X = \{\mu\check{\zeta}^2 + \check{\eta}^2 = 0\}.$$

For  $\mu > 0$ , this equation has no solutions, which corresponds to the characteristic set not touching fiber infinity. For  $\mu \leq 0$ , we have

$$\begin{aligned} \{\langle \xi \rangle^{-2}p = 0\} \cap S^*X &= \Sigma'_+ \cup \Sigma'_-, \\ \Sigma'_\pm &= \{\mu \leq 0, \langle \xi \rangle^{-1} = 0, \check{\zeta} = \mp \operatorname{sgn} r / \sqrt{1 - \mu}, \check{\eta}^2 = -\mu/(1 - \mu)\}, \end{aligned}$$

because  $\check{\zeta}^2 + \check{\eta}^2 = 1$  on  $S^*X$ . In particular, we have

$$L_\pm = \Sigma'_\pm \cap \{\mu = 0\} = \{\mu = 0, \langle \xi \rangle^{-1} = 0, \check{\zeta} = \mp \operatorname{sgn} r, \check{\eta} = 0\}.$$

On  $\{\check{\zeta} \neq 0\} \supset \Sigma'_+ \cup \Sigma'_-$ , we can pass to the system of coordinates

$$(r, \tilde{y}, \tilde{\rho} = |\check{\zeta}|^{-1}, \hat{\eta} = \tilde{\rho}\check{\eta}).$$

Near  $\Sigma'_\pm$  we have  $\zeta = \mp \operatorname{sgn} r \tilde{\rho}^{-1}$  and thus, using  $\partial_\zeta = \pm \operatorname{sgn} r \tilde{\rho}(\tilde{\rho}\partial_{\tilde{\rho}} + \hat{\eta}\partial_{\hat{\eta}})$ ,

$$\tilde{\rho}H_p = \mp 2 \operatorname{sgn} r ((\mu \mp |r|\tilde{\rho})\partial_r - (r \pm \tilde{\rho} \operatorname{sgn} r)(\tilde{\rho}\partial_{\tilde{\rho}} + \hat{\eta}\partial_{\hat{\eta}})) + 2\hat{\eta}\partial_{\tilde{y}}. \quad (\text{B.3})$$

Since  $\mu = \tilde{\rho} = \hat{\eta} = 0$  on  $L_\pm$ , we see that  $L_\pm$  consists of fixed points for  $\tilde{\rho}H_p$ . We also get

$$\tilde{\rho}H_p\mu|_{\Sigma'_\pm} = \pm 4|r|\mu.$$

Therefore, the flow lines on  $\Sigma'_+ \cap \{\mu < 0\}$  go to  $\mu = -\infty$  in the forward direction and to  $L_+$  in the backward direction, while the flow lines on  $\Sigma'_- \cap \{\mu < 0\}$  go to  $\mu = -\infty$  in the backward direction and to  $L_-$  in the forward direction. This is displayed in Figure 6.

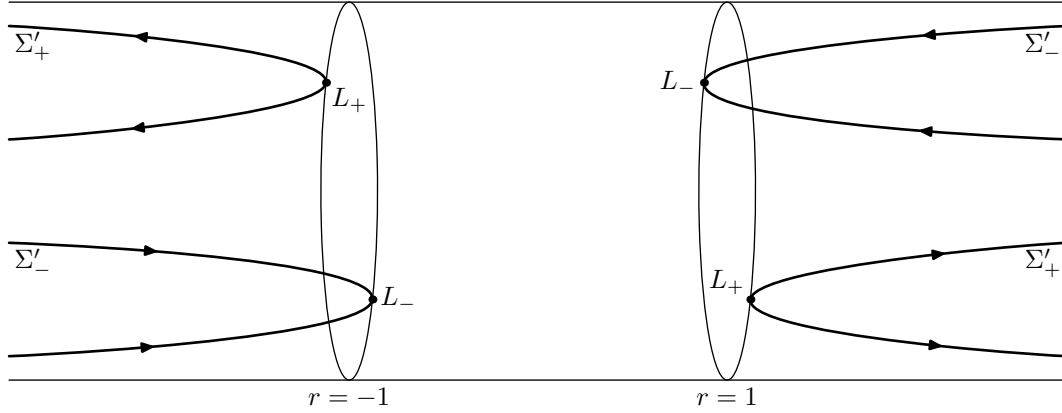


FIGURE 6. Global dynamics of the Hamiltonian flow on the fiber infinity.

We can also use (B.3) to study the dynamics of the flow of  $H_p$  on  $\bar{T}^*M$ , not just on  $S^*M$ , near  $L_{\pm}$ . Omitting the  $\tilde{y}$  variable as the flow does not depend on it, we find that

$$\tilde{\rho}H_p \begin{pmatrix} \mu \\ \tilde{\rho} \\ \hat{\eta} \end{pmatrix} = \begin{pmatrix} \pm 4 & -4 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm 2 \end{pmatrix} \begin{pmatrix} \mu \\ \tilde{\rho} \\ \hat{\eta} \end{pmatrix} + \mathcal{O}(\mu^2 + \tilde{\rho}^2 + \hat{\eta}^2).$$

We see that  $L_+$  is a source and  $L_-$  is a sink. In fact, the eigenvectors of the linearized flow at  $L_{\pm}$  are  $\partial_{\mu}$  with eigenvalue  $\pm 4$ ,  $2\partial_{\mu} \pm \partial_{\tilde{\rho}}$  with eigenvalue  $\pm 2$ , and  $\partial_{\hat{\eta}}$  with eigenvalue  $\pm 2$ . The behavior of the linearized system is pictured on Figure 2 (the horizontal coordinate is  $r$ , and the vertical coordinate  $\tilde{\zeta} = \zeta/\langle\zeta\rangle$  is a compactification of  $\zeta$ ).

We now study the semiclassical behavior, that is, dynamics in the interior  $T^*X$  of  $\bar{T}^*X$ . First of all, we fix  $r$  and study the set of solutions in  $(\zeta, \tilde{\eta})$  to the characteristic equation

$$\mu\zeta^2 + 2r\zeta + \tilde{\eta}^2 - 1 = \mu(\zeta + r/\mu)^2 + \tilde{\eta}^2 - (1 + r^2/\mu) = 0.$$

This is an ellipse when  $\mu > 0$ , a parabola when  $\mu = 0$ , and a hyperbola when  $\mu < 0$ . Therefore, in  $T^*X$  the characteristic set has one connected component for  $\mu \geq 0$  and two components for  $\mu < 0$ ; on the other hand, in  $\bar{T}^*X$  it has one connected component for  $\mu > 0$  and two components for  $\mu \leq 0$ , the additional connected component for  $\mu = 0$  being exactly  $L_-$ , and the intersections of the connected components with  $S^*X$  being exactly  $\Sigma'_{\pm}$ . We then see, as in Figure 7, that the characteristic set  $\{\langle\xi\rangle^{-2}p = 0\} \subset \bar{T}^*X$  can be split into two components  $\Sigma_+$  and  $\Sigma_-$  (the latter consisting of two pieces, corresponding to  $\pm r \geq 1$ ), so that  $\Sigma'_{\pm} = \Sigma_{\pm} \cap S^*X$  and  $\Sigma_- \subset \{\mu \leq 0\}$ . More precisely, we note that near  $\{\mu = 0\}$ , the characteristic set does not intersect the surface

$$\{\tilde{\xi} = -1/2\} = \{\zeta = r\} \subset \bar{T}^*X,$$

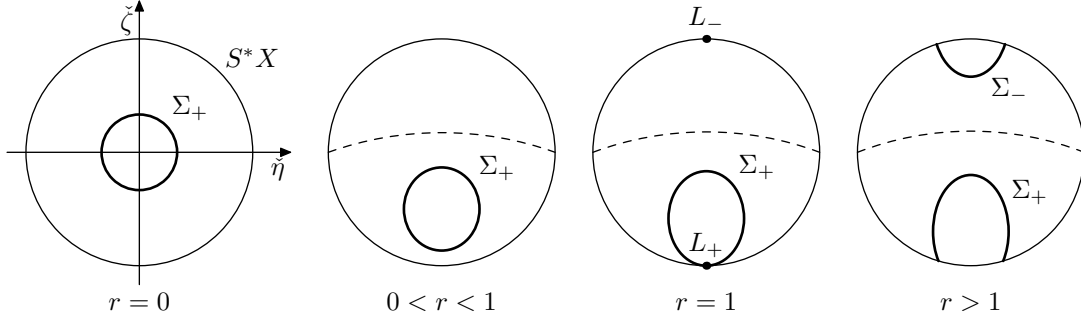


FIGURE 7. The characteristic set in the radial compactified fiber for various values of  $r$ . The dashed line is  $\{\zeta = r\}$ , separating  $\Sigma_+$  from  $\Sigma_-$ .

where  $\tilde{\xi} = -\zeta/(2r)$  is the dual variable to  $\mu$ . Therefore, for  $\varepsilon$  small enough, we can define  $\Sigma_{\pm}$  in  $\{|\mu| < \varepsilon\}$  as

$$\Sigma_{\pm} \cap \{|\mu| < \varepsilon\} = \{\pm(\tilde{\xi} + 1/2) > 0\} \cap \{\langle \xi \rangle^{-2} p = 0\} = \{\mp r(\zeta - r) > 0\} \cap \{\langle \xi \rangle^{-2} p = 0\}.$$

With that definition,  $\Sigma_- \subset \{-\varepsilon < \mu \leq 0\}$  for some  $\varepsilon > 0$ ; hence, we can extend  $\Sigma_{\pm}$  to  $\{\mu > -\varepsilon\}$  by requiring that  $\Sigma_- \cap \{\mu \geq \varepsilon\} = \emptyset$  and  $\Sigma_+ \cap \{\mu \geq \varepsilon\} = \{\langle \xi \rangle^{-2} p = 0\} \cap \{\mu \geq \varepsilon\}$ .

It remains to study the Hamiltonian flow in  $T^*X$ . If  $\mu = 0$ , we have  $H_p \mu = -4r^2 < 0$ ; therefore, the flow lines on  $\Sigma_+ \setminus L_+ = \{\langle \xi \rangle^{-2} p = 0\} \setminus S^*X$  only cross  $\{\mu = 0\}$  in the direction of decreasing  $\mu$ . Together with the behavior of the linearized flow near  $L_{\pm}$  studied before, this gives the behavior of the flow near  $\{\mu = 0\}$ , as in Lemma 4.2. It remains to analyze the behavior of the flow in  $\{\mu > 0\}$ . This can be related to the geodesic flow on the original manifold; we then see that the trapped set corresponds to two trapped trajectories

$$\iota(K) = \{r = \zeta = 0, |\tilde{\eta}| = 1\}$$

while the incoming/outgoing tails (in the sense of (4.8)) are given by

$$\begin{aligned} \iota(\Gamma_+) &= \{\zeta = 0, |\tilde{\eta}| = 1\}, \\ \iota(\Gamma_-) &= \{\zeta = -2r/\mu, |\tilde{\eta}| = 1\}. \end{aligned}$$

Note that  $\iota(\Gamma_+)$  continues smoothly across  $\{\mu = 0\}$  and  $\iota(\Gamma_-)$  converges backwards to  $L_+$ .

All components of Figure 2 are now in place. The horizontal direction corresponds to  $r$  (with  $r$  increasing as we move to the right), with two vertical lines marking  $\partial \overline{M}_{\text{even}} = \{r = \pm 1\}$ , the conformal boundary. The vertical direction corresponds to a compactification of the momentum  $\zeta$ ; the corresponding coordinate  $\tilde{\zeta} = \zeta/\langle \zeta \rangle$  is well-defined away from  $\{\langle \xi \rangle^{-1} = \tilde{\zeta} = 0\}$  on  $\overline{T^*X}$ , and thus on the whole characteristic set. The top and bottom edges of the picture then correspond to  $\{\pm \tilde{\zeta} > 0, \langle \xi \rangle^{-1} = 0\} \subset S^*X$ .

The explicit formulas (B.1) and (B.2) for  $p$  and  $H_p$  allow us to define the escape functions  $f_0$  and  $\hat{f}$ , constructed in the general case in Lemmas 4.7 and 7.1, more directly and explicitly

here. We begin with  $\hat{f}$ , where we will use the fact that no conditions are imposed outside of a small neighborhood of  $\iota(\tilde{K})$ . Observe first that

$$\iota(\tilde{K}) = \{r = \zeta = 0\},$$

so we will be interested in estimates valid for  $r$  and  $\zeta$  sufficiently small. Following [WuZw11, §4.2], [SjZw, §7], [Sj90, §5], let

$$\varphi_+ = \zeta^2, \quad \varphi_- = (\mu\zeta + 2r)^2,$$

be functions measuring the distance squared to  $\iota(\Gamma_+)$  and  $\iota(\Gamma_-)$  respectively. We then have

$$\begin{aligned} H_p \varphi_+ &= 4\zeta^2(r\zeta - 1) = -4\varphi_+(1 + \mathcal{O}(r^2 + \zeta^2)), \\ H_p \varphi_- &= 4(\mu\zeta + 2r)^2(1 - r\zeta) = 4\varphi_-(1 + \mathcal{O}(r^2 + \zeta^2)). \end{aligned}$$

(Near  $\iota(\tilde{K})$  such estimates can be deduced from the hyperbolicity of  $\iota(\tilde{K})$  but in this example we can compute the derivatives directly). Consequently

$$H_p(\varphi_- - \varphi_+) = 4(\varphi_- + \varphi_+) + \mathcal{O}(r^4 + \zeta^4). \quad (\text{B.4})$$

Hence, near  $\iota(\tilde{K})$ ,  $H_p(\varphi_- - \varphi_+) \geq (\varphi_- + \varphi_+)/C$ , which is a nonnegative function vanishing precisely on  $\iota(\tilde{K})$ . To obtain a lower bound of  $1/C$  off a neighborhood of size  $(h/\tilde{h})^{1/2}$  of  $\iota(\tilde{K})$ , as asserted in Lemma 7.1, we take the following ‘logarithmic flattening’ of  $\varphi_- - \varphi_+$ :

$$\hat{f} = \log((h/\tilde{h}) + \varphi_-) - \log((h/\tilde{h}) + \varphi_+).$$

Then, for  $r$  and  $\zeta$  sufficiently small,

$$\begin{aligned} H_p \hat{f} &= 4 \frac{\varphi_-(1 + \mathcal{O}(r^2 + \zeta^2))}{(h/\tilde{h}) + \varphi_-} + 4 \frac{\varphi_+(1 + \mathcal{O}(r^2 + \zeta^2))}{(h/\tilde{h}) + \varphi_+} \\ &\geq 2 \frac{\varphi_-}{(h/\tilde{h}) + \varphi_-} + 2 \frac{\varphi_+}{(h/\tilde{h}) + \varphi_+}, \end{aligned}$$

and this is uniformly bounded from below off of a neighborhood of size  $(h/\tilde{h})^{1/2}$  of  $\iota(\tilde{K})$ . The upper bounds on derivatives in Lemma 7.1 follow from similar arguments.

To construct  $f_0$  we begin with a near-global escape function based on  $\varphi_- - \varphi_+$ :

$$f_{00} = r\zeta + \frac{1}{2}r^2.$$

This function is strictly increasing along flowlines everywhere in  $T^*X \cap \{|\mu| \leq 1\} \setminus \iota(\tilde{K})$ :

$$H_p f_{00} = 2\zeta^2 + 2\mu\zeta r + 2r^2 \geq \zeta^2 + r^2.$$

To obtain  $f_0$  we precompose and multiply by a smooth cutoff as in Lemma 4.7.

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